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Vibration of Rotating Beams with Non-Uniform Cross-Section

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Submitted in Partial Fulfillment of the Requirments for the Degree of Master of Science in Mechanical Engineering
Faculty of Graduate Studies.
University of Jordan.

June / 1995

Committee Decision

This thesis was defended successfully on 21. June 1995

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Dedication

To All Whom I Love and Appreciate:
To My Parents, Brothers
and Sisters

Acknowledgments

I would like to express my deepest gratitude to Dr. Mazen Al-Qaisi for his support and advice through out all stages of this work. Also I would like to thank the members of the mechanical engineering department staff in the university of Jordan whom I am happy and proudful to be one of their students during the under graduate and master programs.

The help of the members of the computer laboratories of the faculty of engineering and technology is highly appreciated.

I am also completely indebted to my family members, especially Ali al shibli, for their constant encouragement and support.

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Nomenclature

$A(x)$	The cross-section area of the beam as a function of the position x.
A_i	The coefficients of the flexural rigidity series
B_i	The coefficients of the mass per unit length series
C_1, C_2, C_3 and C_4	The coefficients of the interpolation function
C_n	The n'th coefficient of the power series used in the solution by series method
E	The modals of elasticity of the beam material
EI_0	The flexural rigidity at the root.
$f(L\xi)$	The non-dimensional flexural rigidity function.
F_c	The centrifugal force.
$g(L\xi)$	The non-dimensional mass per unit length function.
h	The finite element length.
i and j	Indices.
$I(x)$	Second moment of area of the beam cross-section
j	Infix
$K_{i,j}$	The i,j element of the finite element stiffness matrix
K_R	Rotational stiffness.
K_T	Translation stiffness.
L	The beam length.
m	The upper limit of the flexural rigidity and mass per unit length series.

$M_{i,j}$	Th i,j element of the finite element mass matrix
N	The number of finite elements.
n	Infix
P_1, P_2, P_3 and P_4	The internal forces on the finite element boundaries
r	Root offset radius.
T	Total Kinetic energy.
U	Total potential energy.
u	Axial deflection of the beam cross-section
u_x	The partial derivative of the axial deflection of the beam with respect to x
V	The velocity of the cross section of the beam at x position also the trial function used in the finite element method formulation
v	Dummy variable.
V_c	The potential energy of the centrifugal force
V_L	The velocity of the tip of the beam.
w	The deflection of the beam as a function of time and position.
W(0)	The deflection of the beam at the root.
W(x)	The shape function which describes the motion of the beam also the interpolation function used in the finite element method.
W_1, W_2, W_3 and W_4	The nodal values of the finite element
W_{ex}	The work of the external forces.
w_t	The first partial derivative of the deflection function with respect to t.
w_{tt}	The second partial derivative of the deflection

	function with respect to t.
w_x	The first partial derivative of the deflection function with respect to x.
$W_x(0)$	The sloop of the beam at the root.
w_{xx}	The second partial derivative of the deflection function with respect to x.

Greek Symobls

α	The root offset ratio.
β	The translation stiffness ratio.
γ	The rotation stiffness ratio.
η	The non-dimensional rotational speed.
ρ	The beam material density.
θ	The setting angle of the beam.
μ	The tip mass ratio.
Ω	The rotational speed of the beam.
λ	The eigenvalue of the eigen value problem.
ω	The natural frequency.
ξ	The beam non-dimensional coordinate.
δ	The variation symbol.
ρA_0	The mass per unit length of the beam material at the root.
ξ_n	The starting node coordenat for the n'th finite element
ξ_{n+1}	The End node coordenat for the n'th finite element

Abbreviations

EPS	The solution convergence tolerance
FORD	The approximating function order

Abstract

Vibration of Rotating Beams with Non-Uniform Cross-Section

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In this research the problem of free vibration of rotating beams with non-uniform cross-section was studied using the power series method. The general case of rotating beams with root offset radius, setting angle and tip mass with flexible restraint to the root was considered. The simple Euler beam model was used and the Hamilton's principle was employed to derive the equation of motion and the boundary conditions. Then the equation of motion and boundary conditions were reduced to non-dimensional form.

The system governing differential equation was solved using the power series method and the finite element method for the case where the cross-section properties variation are given in a power series form. The results obtained were compared with some exact results found in the literature. The finite element solution was used to compare the power series solution for the cases for which no solution is available. For an exponentially

solution for the cases for which no solution is available. For an exponentially varying cross-section properties a parameteric study was conducted . In this study the effects of the rotational speed, the tip mass, the root offset radius, the setting angle and the root flexibility were studied.

It was found that the increase in the rotational speed, the root offset radius, the translation stiffness and the rotational stiffness at the root increases the natural frequencies. Also the increase in the setting angle and the tip mass decreases the natural frequencies obtained. It was also found that the finite element solution shows excellent agreement with the power series solution. It was concluded that power series method is a very powerful method if computerized.

Chapter One

Literature Survey

1.1 Introduction

The problem of free vibration of rotating beams has received attention in the literature because of its importance in design and vibration, it has applications in the design of many mechanical systems like the helicopter rotor blades, wind turbine blades and flexible satellite booms.

The problem of free vibration of a rotating beam leads to a differential equation with variable coefficients for both uniform cross-section and non-uniform cross-section cases, therefore; the solution procedure which can be used for rotating beams with uniform cross-section can be extended to the case of non-uniform cross-section beams with a little extra effort.

The parameters affecting the natural frequencies are the boundary conditions which includes the supporting method at the root and the tip mass if exists, also the rotational speed, setting angle and the root offset radius.

In the sections follow a review for some of the papers available in the literature is presented with special consideration for the solution procedure used, then comments on these works are made .

1.2 Papers Review

Many papers available in the literature that deal with rotating beams with uniform cross-section or non-uniform cross-section variations. These works have used different solution procedures to solve for different boundary conditions, setting angles, root offset radius and rotating speeds.

Du, Lim and Liew [1] used Timoshenko beam theory to model a thick rotating, uniform cross-section beam with tip mass, root offset radius and a setting angle of 90 degrees. The resulting equation was solved by using the series method. The effect of the root offset radius and tip mass were studied . The results found didn't show much difference compared with the works using Euler beam model.

Fox and Burdess [2] solved the problem of vibration of rotating beams with uniform cross-section in the case where the root offset from the spin axis such that part of the beam is directed inwards towards the axis of rotation. The problem was solved using Galerkin's method. It was found that the fundamental natural frequency is a function of the rotational speed and the functional relationship depends on the ratio of the root offset to the beam length.

Hao [3] investigated the problem of rotating, uniform cross-section beam with a tip mass using the finite element method. The effect of the root offset radius and tip mass were incorporated into the finite element.

Hodges [4] used a semi-empirical method involving asymptotic expansion to obtain an approximate formula for the fundamental natural frequency of a uniform cross-section beam fixed out of the axis of rotation. The results were found to have an excellent agreement with results from previous works.

Kammer and Schlack [5] studied the effects of non-constant angular velocity upon the vibration of uniform cross-section, rotating, Euler beam using a perturbation technique called the KBM method . It was used to derive approximated solutions and expressions for the boundaries between stable and unstable motion.

Khulief [6] investigated the natural frequencies of a rotating, tapered cross-section beam with tip mass by using the finite element method. The finite element mass and stiffness matrices were derived by using consistent mass and stiffness formulation. The beam was assumed to be linearly tapered in two planes . The generalized eigenvalue problem was solved for a wide range of rotational speed and tip mass variations, both hinged and fixed end conditions were considered .

Krishna Hurly and Romon [7] studied nonlinear natural vibration characteristics and the dynamic response of fixed and fully articulated rotor blades of rectangular cross section using the finite element method . The results show the effect of geometrical non-linearity on the natural frequencies .

Lee and Kuo [8] studied the upper and lower bounds of the fundamental natural frequency of a rotating beam with an elastic restrained root by using Rayleigh's and minimum energy principles. The influence of hub radius, setting angle, angular rotational speed and elastic restraints on the fundamental natural frequency were studied by using transfer matrix method.

Low [9] used a method of eigen analysis to analyze a system consists of a rotating, uniform cross-section beam. The resulting system of equations was solved using the secant method.

Murthy [10] used the transmission matrix method to determine the natural frequencies and mode shapes for rotating blades which were modeled as a simple Euler beam.

Putter and Manor [11] studied the natural frequencies of tapered, radial beam mounted on a rotating disc at 90 degrees setting angle by using the finite element method. They used fifth order polynomial to describe the deflection within the finite element.

Rouch and Koa [12] formulated the stiffness, mass and gyroscopic matrices of a rotating beam by using a cubic polynomial for the transverse displacement. The effects of shear and cross-sectional rotation were included to provide a Timoshenko beam formulation. The method was applied to a circular ,linearly tapered beam .

Storti and Aboelange [13] studied the vibration of a tapered beam as a model for a rotating turbine blade using hyper geometric functions. The problem was solved for different values of setting angle and rotational speed.

Subrahmanyam and Kaza [14] studied the effects of pre-twist, pre-cone, setting angle and coriolis forces on the vibration and buckling behavior of rotating, uniform cross-section beams. In this study the beam was considered to be clamped on the axis of rotation in one case and off the axis of rotation in the other. Two methods are employed for the solution one based upon the finite difference approach and the other based upon the minimum potential energy functional with a Ritz type solution. The effects of the pre-twist, pre-con, setting angle, thickness ratio and Coriolis forces on the natural frequencies were also studied.

Wright, Smith, Threshar and Wange [15] used the method of Frobenius to solve for the exact frequencies and mode shapes for a rotating beam with both the flexural rigidity and the mass per unit length distribution varying linearly. They solved for uniform cross-section and tapered cross-section beams with root offset, setting angle and tip mass for both hinged and fixed root boundary condition .

1.3 Comments

The works available in the literature dealing with the problem of vibration of rotating beams can be classified into two sets. The first which deals with uniform cross-section beams and the second is that which deals with non-uniform cross-section beams.

For the case of uniform cross-section beams the effects of rotational speed, setting angle, tip mass, radius of the root offset and the root attachment method were studied using different solution procedures. Reference[1] studied the vibration of rotating, uniform cross-section beam which was modeled as a Timoshenko beam by using the series method. Reference [2] used Galerkin's method. Reference [3] studied this problem using the finite element method. Reference [4] used a semi-empirical method to obtain an approximate formula for the fundamental natural frequency. In reference [5] the uniform cross-section beam was studied using a method called KBM. Reference [8] used Rayleigh's and the minimum energy principle. Reference [9] solved this problem using eigen analysis. Reference [14] used the finite difference method and the minimum potential energy principle in solving this problem. Reference [15] solved for the exact natural frequencies using the method of Frobenius (series method).

For the case of non-uniform cross-section beams the only case which was studied is the tapered cross-section beam. References [6], [7], [11] and [12] studied the problem of free vibration of rotating, linearly tapered beams using the finite element method. Reference [10] used the transmission matrix method .. In Reference [13] hyper geometric functions were used to study a rotating, linearly tapered cross-section beam. Reference [15] used the method of Frobenius (series method) to study the vibration of linearly tapered cross-section beams.

From this we note that the problem of rotating, uniform cross-section beam case has been studied extensively. But for rotating beams with non-uniform cross-section the only case which has been studied is that of a linearly tapered cross-section beam. Most of the studies used approximate methods to solve this problem like the finite element method, Ritz method or Rayleigh's method, which do not give exact solutions. The methods which were used to solve the problem of uniform cross-section beams can be extended in some way to solve for non-uniform cross-section beams, most of those methods are also approximate methods. The parameters which affect the vibration of non-uniform cross-section beams are those which affect the vibration of uniform cross-section beams, which were studied for the case of linearly tapered cross-section beams only. The only method which can solve this problem exactly is the method of Frobenius (power series) method which is the standard analytical method for solving ordinary differential equations with variable coefficients. It was used to solve for the uniform cross-section and tapered cross-section beams in reference [15] and for the uniform cross-section beam which was modeled as a Timoshenko beam in reference [1].

In this research the method of Frobenius (power series method) will be used to solve a general case of rotating beam with non-uniform cross-section properties variation which its properties are given in power series form. For an exponentially varying cross-section properties the effect of the tip mass, root offset radius, angular speed, setting angle and boundary conditions will be studied . To chick the power series solution the same problem will be solved by the finite element method and some of the results will be compared with that found from the Frobenius (power series) method.

Chapter Two

Formulation of the Problem of Free Vibration of Rotating Beams with Non-Uniform Cross-Section

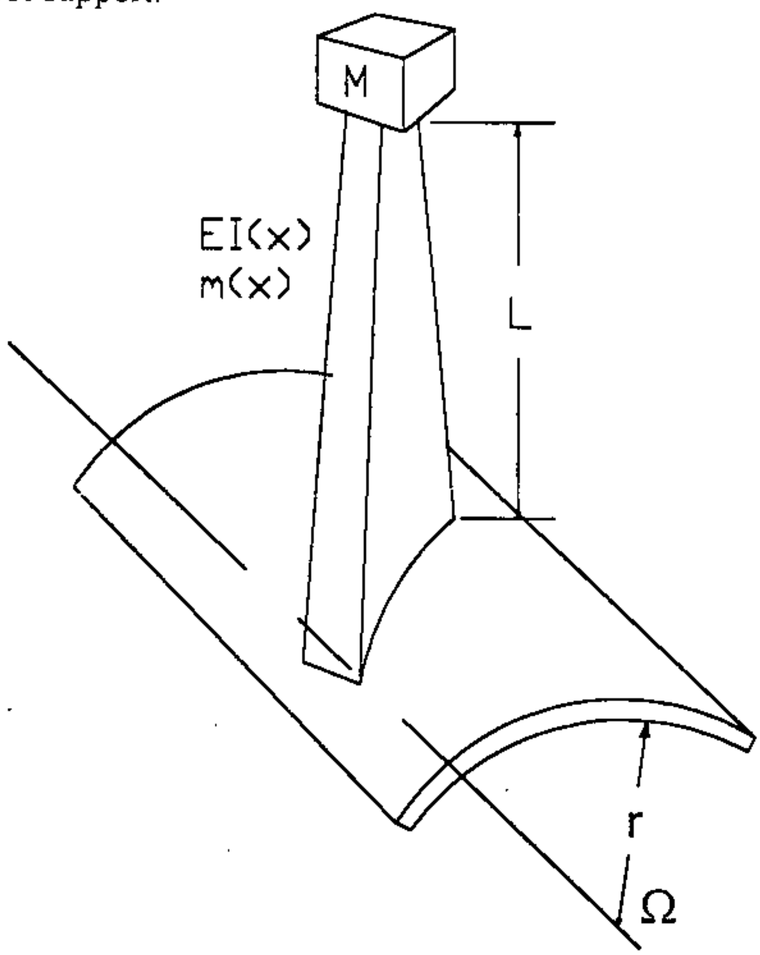
2.1 Introduction

In this chapter the equation of motion of rotating beams with non-uniform cross-section is derived using the Hamilton's principle. The Eulers beam model will be employed to formulate the system equation and the boundary conditions. All the assumptions used in the modeling are also stated. By assuming free vibration the partial differential equation with its boundary conditions will be written as an ordinary differential equation . Finally in section (2.3) the equation of motion and the boundary conditions will be reduced to non-dimensional form. The non-dimensional sets will be defined such that the special case of non-rotating beams will be included as a special case of the rotating beam with zero rotational speed .

2.2 Derivation of the Equation of Motion Using the Hamilton's Principle

The first step in the solution of any vibration problem is to derive its equations of motion. This section deals with the derivation of the equation of motion governing the problem of free vibration of rotating beams with non-uniform cross-section, subjected to various boundary conditions which model different practical conditions at the boundaries.

Consider the general case of a non-uniform cross-section beam shown in figure (2.1). The proposed beam is shown in figure (2.2). In the proposed model the beam is assumed to be of length (L), attached to a hub of radius (r), at setting angle (θ) and rotates with a constant speed (Ω) .Also the beam is assumed to have a tip mass (M) with elastic restraint to the hub at the root. The translation stiffness is K_T and the rotational stiffness is K_R of the root support.



Fig(2.1)

In the derivation of the equation of motion the Hamilton's principle will be used because of its simplicity and advantages; which are :-

1- It requires writing simple quantities, the kinetic and potential energies of the system .

2- This method gives the boundary conditions and the equation of motion simultaneously .

The Hamilton's principle states that " of all the paths of a motion of a system during any arbitrary interval of time t_0 to t_1 , the actual path will be that one for which the integral :-

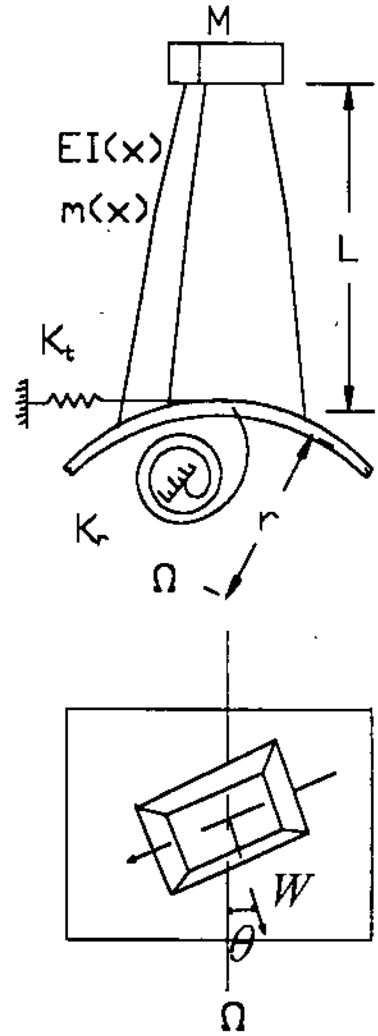
$$\int_{t_0}^{t_1} \delta(W_{ex} - U + T)dt \qquad \text{.......................} \quad (2.1)$$

where :- W_{ex} : The work done by external forces .

U :The potential energy of the system .

T : The kinetic energy of the system .

has a stationary value ". It can be shown that this stationary value will be the minimum of this integral .



Fig(2.2)

To derive the equation of motion the Euler Beam model will be employed . The assumptions in this model are:-

1- The cross-section dimensions are small relative to the beam length, so that the shear energy is negligible.

2- The vibration of the beam is in plane vibration and the out-off plane vibration is negligible .

3- The center line of the beam doesn't change length upon deformation

4- The change in the potential energy due to the force of gravity is considered to be negligible .

5- The kinetic energy due to the rotation of beam cross-section and axial deformation are negligible .

6- The Coriolus acceleration is assumed to be negligible .

Under the consideration of these assumptions it is required to write expressions for the work of the external forces (W_{ex}), the potential energy (U) and the kinetic energy (T). Since the case of free vibration is considered the work of the external forces (W_{ex}) is zero.

The total kinetic energy of the system is given by :

$$T = \frac{1}{2}\int_0^L \rho A(x) V^2 dx + \frac{M}{2} V^2(L) + \frac{J}{2}(\Omega \sin(\theta) + w_{xt})^2 \quad \text{..............} \quad (2.2)$$

where : ρ : The density of the beam material .

A(x) : The cross-section area of the beam at x .

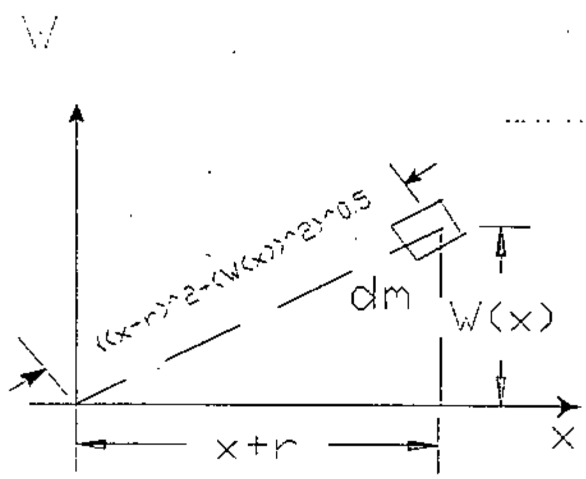
V : The velocity of the beam cross-section as a function of x

V_L : The velocity of the beam cross-section at the tip (x=L).

$w_{xt}(L)$: The second partial derevative of the deflection function with respect to time and position

J : The rotational inertia of the tip mass.

The velocity (V) consists of the component of the rotational velocity in the direction of vibration in addition to that due to the vibration itself. If we consider a differential mass element at position x from the root as shown in figure (2.3). The velocity of the differential mass element is given by :-



Fig(2.3)

$$V = w_t + \sqrt{(r+x)^2 + w^2}\ \Omega \sin(\theta) \qquad \text{.................} \quad (2.3)$$

where w_t is the partial derivative of the deflection w with respect to time.

and for the tip at **x=L**, the velocity is:-

$$V(L) = w_t(L) + \sqrt{(r+L)^2 + w^2(L)}\ \Omega \sin(\theta) \qquad \text{.................} \quad (2.4)$$

Substituting eq(2.3) and eq(2.4) into eq(2.2) yields :-

$$T = \frac{1}{2}\int_0^L \rho A(x)(w_t + \sqrt{(r+x)^2 + w^2}\ \Omega\sin(\theta))^2 dx + \frac{M}{2}(w_t(L) + \sqrt{(r+L)^2 + w^2(L)}\ \Omega\sin(\theta))^2 + \frac{J}{2}(\Omega\sin(\theta) + w_{xt})^2 \qquad \text{..............} \quad (2.5)$$

to avoid non-linearity this equation may be simplified to :-

$$T = \frac{1}{2}\int_0^L \rho A(x)(w_t^2 + \left((r+x)^2 + w^2\right)\Omega^2\sin^2(\theta))dx + \frac{M}{2}(w_t^2(L) + \left((r+L)^2 + w^2(L)\right)\Omega^2\sin^2(\theta)) + \frac{J}{2}(\Omega\sin(\theta) + w_{xt})^2 \qquad \text{............} \quad (2.6)$$

Before going forward to write the total potential energy. It is required to write an expression for the potential energy due to the centrifugal force. If **Fc** is the centrifugal force and the beam deflects an amount **du** in the axial direction, where **u** is the axial displacement of the cross-section at position x, which is a function of time and position. The work done by the centrifugal force is given by

$$dW = F_c du = F_c \frac{\partial u}{\partial x} dx$$
$$= F_c u_x \, dx$$

but from basic principles in dynamics. it is known that the change in the work is the negative of the change of the potential energy; i.e.:-

$$dW = -dV$$

using this relation with the above one. after integrating the resulting expression with respect to x from zero to L. and by assuming that at the root (x=0) the potential energy is zero (V=0). The potential energy due to the centrifugal force becomes :-

$$V_c = -\int_0^L F_c u_x dx$$

The total potential energy consists of that due to the bending strain energy and the centrifugal force in addition to that due to the flexible restraints at the root. From this the total potential energy can be written as :

$$U = \frac{1}{2}\int_0^L EI(x) w_{xx}^2 dx - \int_0^L F_c u_x dx + \frac{1}{2} K_T W^2(0) + \frac{1}{2} K_R w_x^2(0) \qquad \text{............(2.7)}$$

where :- . EI(x) : The flexural rigidity of the beam cross section at position x

c : The centrifugal force due to the inertia of the part of the beam beyond the position x .

K_T : The translation stiffness at the root .

K_R : The rotational stiffness at the root .

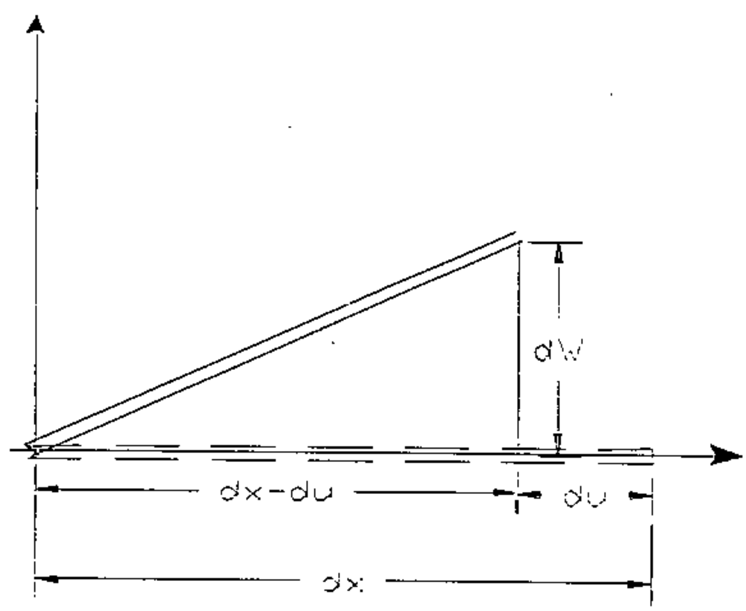
The centrifugal force is given by :-

$$F_c = \Omega^2 [\int_x^L \rho A(v)(r+v) dv + M(r+L)] \qquad \text{...............(2.8)}$$

substituting eq(2.8) into eq(2.7), yields :-

$$U = \frac{1}{2}\int_0^L EI(x)w_{xx}^2dx - \int_0^L \Omega^2[\int_x^L \rho A(v)(r+v)dv + M(r+L)]u_x dx + \frac{1}{2}K_T w^2(0) + \frac{1}{2}K_R w_x^2(0) \qquad \text{.......... (2.9)}$$

From eq(2.9) the necessity to have a relation that relates the axial displacement to the transverse deflection is clear. From the assumption that the length of the center line of the beam doesn't change upon deformation, if a differential element of length dx is considered before and after deformation as shown in fig(2.4). one can write :



Fig(2.4)

$$(dx)^2 = (dw)^2 + (dx+du)^2$$

simplifying and dividing by $(dx)^2$:-

$$\left(\frac{dw}{dx}\right)^2 = -2\frac{du}{dx} - \left(\frac{du}{dx}\right)^2$$

Since $(\partial u/\partial x)$ is small. Its square is very small so it can be negleced. This leads to the simple relation :-

$$\frac{du}{dx} = -\frac{1}{2}\left(\frac{dw}{dx}\right)^2 \qquad \text{.............. (2.10)}$$

substituting eq(2.10) into eq(2.9). The total potential energy becomes :-

$$U = \frac{1}{2}\int_0^L EI(x)w_{xx}^2 dx + \frac{1}{2}\int_0^L \Omega^2 [\int_x^L \rho A(v)(r+v)dv + M(r+L)]\, w_x^2 dx + \frac{1}{2}K_T w^2(0) + \frac{1}{2}K_R w_x^2(0)$$

...........(2.11)

From eq(2.6) the variation in the kinetic energy is :-

$$T = \int_0^L \rho A(x)(w_t\delta w_t + w\Omega^2 \sin^2(\theta)\delta w)dx + M(w_t(L)\delta w_t(L) + w(L)\Omega^2 \sin^2(\theta)\delta w(L)) + J(\Omega\sin(\theta) + w_{xt})\delta w_{xt} \qquad \text{.............. (2.12)}$$

Taking the integration of δT with respect to time over the interval [t_0 , t_1] one gets :-

$$\int_{t_0}^{t_1}\delta T dt = \int_{t_0}^{t_1}\int_0^L \rho A(x)w_t\delta w_t dxdt + \int_{t_0}^{t_1}\int_0^L \rho A(x)\Omega^2 \sin^2(\theta)w\delta w dxdt + \int_{t_0}^{t_1} Mw_t(L)\delta w_t(L)dt + \int_{t_0}^{t_1} M\Omega^2 \sin^2(\theta)w(L)\delta w(L)dt + \int_{t_0}^{t_1} J(\Omega + w_{xt}(L))\delta w_{xt}(L)\, dt$$

.... (2.13)

integrating the first and third terms by parts with respect to time :-

$$\int_{t_0}^{t_1}\delta T dt = \int_0^L \rho A(x)w_t\delta w dx \Big|_{t_0}^{t_1} - \int_{t_0}^{t_1}\int_0^L \rho A(x)w_{tt}\delta w dxdt + \int_{t_0}^{t_1}\int_0^L \rho A(x)\Omega^2 \sin^2(\theta)w\delta w dxdt + Mw_t(L)\delta w(L)\Big|_{t_0}^{t_1} - \int_{t_0}^{t_1} Mw_{tt}(L)\delta w(L)dt + \int_{t_0}^{t_1} M\Omega^2 \sin^2(\theta)w(L)\delta w(L)dt + J(\Omega\sin(\theta) + w_{xt}(L)\delta w_x(L)\Big|_{t_0}^{t_1} - \int_{t_0}^{t_1} J\frac{\partial}{\partial t}((\Omega\sin(\theta) + w_{xt}(L)\delta w_x(L))dt \qquad \text{.............. (2.14)}$$

But from the theory ofthe Hamilton's principle, the first, fourth and seventh terare zeros. After rearranging the remaining terms the variation in the kinetic energy becomes :-

$$\int_{t_0}^{t_1} \delta T dt = -\int_{t_0}^{t_1}\int_0^L \rho A(x)\left(w_{tt} - \Omega^2 \sin^2(\theta) w\right)\delta w dx dt -$$
$$\int_0^L M\left(w_{tt}(L) - \Omega^2 \sin^2(\theta) w(L)\right)\delta w(L) dx - \int_{t_0}^{t_1} J w_{xtt}(L)\delta w_x(L) dt \quad \text{.............. (2.15)}$$

Also the variation in the potential energy from eq(2.11) is :-

$$\delta U = \int_0^L EI(x) w_{xx}\delta w_{xx} dx + \int_0^L \Omega^2 [\int_x^L \rho A(v)(r+v) dv + M(r+L)] w_x \delta w_x dx +$$
$$K_T w(0)\delta w(0) + K_R w_x(0)\delta w_x(0)$$

.... (2.16)

taking the integration of δU with respect to time over the interval $[t_0, t_1]$, after integrating by parts the first term twice with respect to x and the second once with respect to x. The variation in the potential energy eq(2.16) reduced to :-

$$\int_{t_0}^{t_1} \delta U dt = \int_{t_0}^{t_1} EI(x) w_{xx}\delta w_x \Big|_0^L dt - \int_{t_0}^{t_1} \left(EI(x) w_{xx}\right)_x \Big|_0^L \delta w dt + \int_{t_0}^{t_1}\int_0^L \left(EI(x) w_{xx}\right)_{xx} \delta w dx dt +$$
$$\int_{t_0}^{t_1} [\int_x^L \rho A(v)(r+v) dv + M(r+L)]\Omega^2 w_x \delta w \Big|_0^L dt - \int_{t_0}^{t_1}\left([\int_x^L \rho A(v)(r+v) dv + M(r+L)] w_x\right)_x \Omega^2 \delta w dx dt +$$
$$\int_{t_0}^{t_1} K_T w(0)\delta w(0) dt + \int_{t_0}^{t_1} K_R w_x(0)\delta w_x(0) dt$$

.... (2.17)

From the Hamilto's principle eq(2.1). Substituting eq(2.14) and eq(2.17) in eq(2.1), yields:-

$$\int_{t_0}^{t_1}\delta(W_{ex}-V+T)dt=-\int_{t_0}^{t_1}\int_0^L \rho A(x)\left(w_{tt}-\Omega^2\sin^2(\theta)w\right)\delta w\,dx\,dt-$$

$$\int_0^L M\left(w_{tt}(L)-\Omega^2\sin^2(\theta)w(L)\right)\delta w(L)dx$$

$$-\int_{t_0}^{t_1} EI(x)w_{xx}\delta w_x\Big|_0^L dt+\int_{t_0}^{t_1}\left(EI(x)w_{xx}\right)_x\Big|_0^L\delta w\,dt\ -\int_{t_0}^{t_1}\int_0^L\left(EI(x)w_{xx}\right)_{xx}\delta w\,dx\,dt-$$

$$\int_{t_0}^{t_1}\Big[\Big(\int_x^L \rho A(v)(r+v)dv+M(r+L)\Big)\Omega^2 w_x\delta w\Big|_0^L dt+\int_{t_0}^{t_1}\int_0^L\left(\Big(\int_x^L\rho A(v)(r+v)dv+M(r+L)\Big)w_x\right)_x\Omega^2\delta w\,dx\,dt$$

$$-\int_{t_0}^{t_1}K_T w(0)\delta w(0)dt\ -\int_{t_0}^{t_1}K_R w_x(0)\delta w_x(0)dt=0 \qquad \text{.............. (2.18)}$$

collecting terms, and simplifying :-

$$-\int_{t_0}^{t_1}\int_0^L\left(\left(EI(x)w_{xx}\right)_{xx}-\left(\Big(\int_x^L\rho A(v)(r+v)dv+M(r+L)\Big)w_x\right)_x+\rho A(x)(w_{tt}-\Omega^2\sin^2(\theta)w)\right)\delta w\,dx\,dt\ -$$

$$\int_{t_0}^{t_1}\left(\left(EI(x)w_{xx}\right)_x-\Big(\int_x^L\rho A(v)(r+v)dv+M(r+L)\Big)\Omega^2 w_x+K_T W\right)_{x=0}\delta w(0)dt\ +$$

$$\int_{t_0}^{t_1}\left(\left(EI(x)w_{xx}\right)_x-\Big(\int_x^L\rho A(v)(r+v)dv+M(r+L)\Big)\Omega^2 w_x-M(w_{tt}-\Omega^2\sin^2(\theta)w)\right)_{x=L}\delta w(L)dt\ -$$

$$\int_{t_0}^{t_1}\left(EI(x)w_{xx}\right)_{x=L}\delta w_x(L)dt+\int_{t_0}^{t_1}\left(\left(EI(x)w_{xx}\right)-K_R w_x\right)_{x=0}\delta w_x(0)dt\ =\ 0$$

.... (2.19)

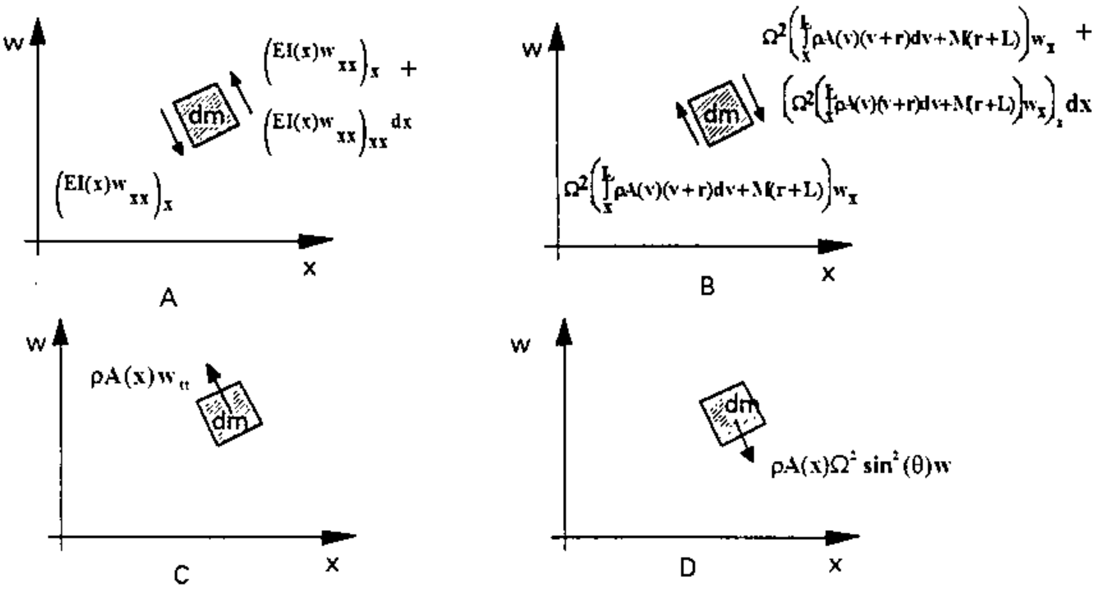
To satisfy eq(2.19) each integral in this equation should equal zero .The first integral will give the equation of motion for the system and the remaining integrals will give the boundary conditions, Hence :-

$$\left(\left(EI(x)w_{xx}\right)_{xx}-\Omega^2\left(\Big(\int_x^L\rho A(v)(r+v)dv+M(r+L)\Big)w_x\right)_x+\rho A(x)(w_{tt}-\Omega^2\sin^2(\theta)w)\right)\delta w=0$$

But it is Known that $\delta w\neq 0$. This means that the term between the brackets should equal zero :-

$$\left(EI(x)w_{,xx}\right)_{,xx} - \Omega^2\left(\left(\int_x^L \rho A(v)(r+v)dv + M(r+L)\right)w_{,x}\right)_{,x} + \rho A(x)\left(w_{,tt} - \Omega^2 \sin^2(\theta)w\right) = 0 \qquad \text{..............} (2.20)$$

which is the equation of motion for the system.

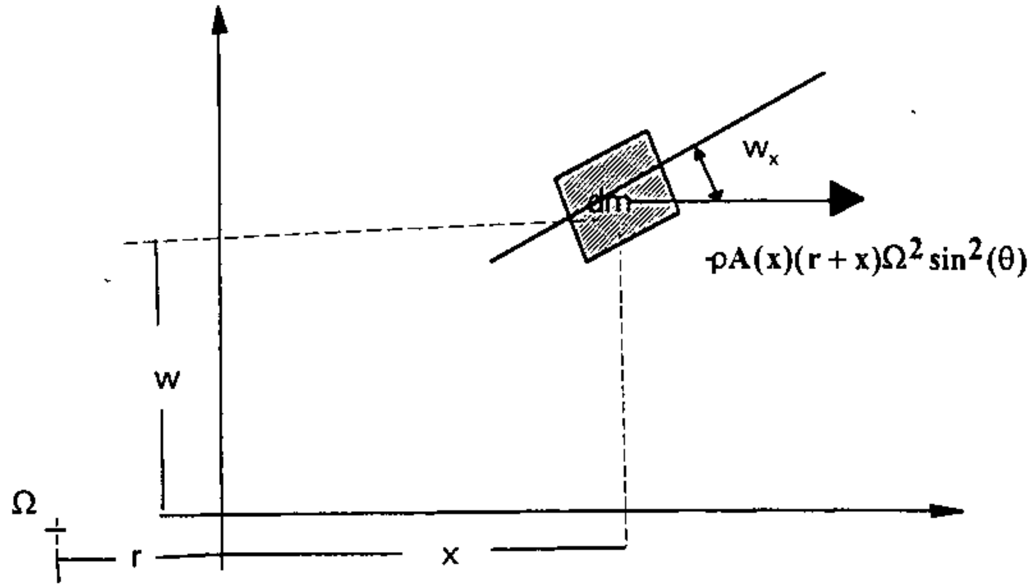


Fig(2.5)

Figure (2.5) show the physical meaning of the terms of the system equation. Considering a differential mass element dm. in terms of the beam material and the cross-section area, if the length of the mass element is dx then the mass of the element is given by :-

$$dm = \rho A(x)dx$$

It is clear from figure (2.5.A) that the first term in the system equation eq(2.20) is the change in the internal force. Also the second term represents the change in the centrifugal force in the direction of vibration. the first part of the last term in eq(2.20) is the inertia of the mass element dm due to vibration as shown in figure (2.5.C). The second part in the last term is the component of the inertia force of the mass element due to the rotation of the hub in the direction of vibration as shown in figure (2.5.D).



Fig(2.6)

To explain the meaning of second part of the last term in eq (2.20) consider figure (2.6) which shows the inertia force due to the rotation of the hub for the differential mass element dm. From figure (2.6) the component of the inertia force due to the rotation of the hub in the direction of vibration is given by:-

$$F_{IR} = \rho A(x)(x+r)\Omega^2 \sin^2(\theta) w_x$$

If the slop (w_x) is approximated by :-

$$w_x = \frac{w}{(x+r)}$$

using this relation in the above one. One gets the second part in the last term in the system equation which is :-

$$F_{IR} = \rho A(x)\Omega^2 \sin^2(\theta) w$$

For the rest of the integrals one may have :-

1- The second and the fifth integrals in eq(2.19) give the terms :-

$$\left((EI(X)w_{xx})_x - \left(\int_x^L \rho A(v)(r+v)dv + M(r+L)\right)\Omega^2 w_x + K_T w\right)_{x=0} \delta w(0) = 0$$

$$\left((EI(x)w_{xx}) - K_R w_x\right)_{x=0} \delta w_x(0) = 0$$

these two equations give the boundary conditions at the root (x=0). The following cases can be considered :-

a - For fixed root $\delta w(0)$ and $\delta w_X(0)$ are zeroes. This means that $w(0)$ and $w_X(0)$ are constants, but for the beam considered both $w(0)$ and $w_X(0)$ are zeros, hence :-

$$w(0) = 0 \qquad \text{.................. (2.21)}$$

$$w_x(0) = 0 \qquad \text{................. (2.22)}$$

b- For hinged root $\delta w(0)$ and K_R are zeroes, but $\delta w_X(0)$ is not zero because the beam can rotate freely at the root, this means that the conditions at the root become :-

$$w(0) = 0 \qquad \text{.............. (2.23)}$$

$$\left(EI(x)w_{xx}\right)_{x=0} = 0 \qquad \text{.............. (2.24)}$$

c- Finally if the case of flexible attachment at the root is considered. $\delta w(0)$, $\delta w_X(0)$, K_R and K_T are all non-zeros which means that :-

$$\left(\begin{array}{l}(EI(X)w_{xx})_x - (\int_x^L \rho A(v)(r+v)dv + \\ M(r+L))\Omega^2 w_x + K_T w\end{array}\right)_{x=0} = 0 \qquad \text{.............. (2.25)}$$

$$\left((EI(x)w_{xx}) - K_R w_x\right)_{x=0} = 0 \qquad \text{.............. (2.26)}$$

2- The third and fourth integrals in eq(2.19) for the tip gives :-

$$\left((EI(X)w_{xx})_x - (\int_x^L \rho A(v)(r+v)dv + M(r+L))\Omega^2 w_x - M(w_{tt} - \Omega^2 \sin^2(\theta) w)\right)_{x=L} \delta w(L) = 0$$

$$\left(EI(x)w_x + Jw_{xtt}\right)_{x=L} \delta w_x(L) = 0$$

but since the tip of the beam is free this means that $\delta w(L)$ and $\delta w_X(L)$ are not zeroes which leads to the boundary conditions :-

$$\left((EI(X)w_{xx})_x - (\int_x^L \rho A(v)(r+v)dv + M(r+L))\Omega^2 w_x - M(w_{tt} - \Omega^2 \sin^2(\theta) w) \right)_{x=L} = 0$$

.... (2.27)

$$\left(EI(x)w_{xx} + Jw_{xtt} \right)_{x=L} = 0 \qquad \text{................ (2.28)}$$

For free vibration the time dependance is harmonic and has the form:-

$$:- \; w(x,t) = W(x)\, e^{i\omega t} \qquad \text{............... (2.29)}$$

where W(x) : A function which describe the shape of the motion

ω :The natural frequency .

if eq(2.29) is substituted in eq(2.20) the equation of motion becomes :-

$$\frac{d^2}{dx^2}\left(EI(x)\frac{d^2W}{dx^2} \right) - \Omega^2 \frac{d}{dx}\left((\int_x^L \rho A(v)(r+v)dv + M(r+L))\frac{dW}{dx} \right) - \qquad \text{........... (2.30)}$$

$$\rho A(x)(\omega^2 + \Omega^2 \sin^2(\theta))W = 0$$

and for the boundary conditions :-

1: at the root (x=0):

a- Fixed root case :-

$$W(0) = 0 \qquad \text{............... (2.31)}$$

$$\left. \frac{dW}{dx} \right|_{x=0} = 0 \qquad \text{............... (2.32)}$$

b- For the hinged case :-

$$W(0) = 0 \qquad \text{............... (2.33)}$$

$$\left(EI(x)\frac{d^2W}{dx^2} \right)_{x=0} = 0 \qquad \text{............... (2.34)}$$

c - For the flexible attachment case :-

$$\left(\frac{d}{dx}(EI(x)\frac{d^2W}{dx^2}) - (\int_x^L \rho A(v)(r+v)dv + M(r+L))\Omega^2 \frac{dW}{dx} + K_T W \right)_{x=0} = 0$$

.... (2.35)

$$\left((EI(x)\frac{d^2W}{dx^2}) - K_R \frac{dW}{dx} \right)_{x=0} = 0 \qquad \text{............... (2.36)}$$

2- For the tip (x=L)

$$\left(\frac{d}{dx}(EI(X)\frac{d^2W}{dx^2}) - (\int_x^L \rho A(v)(r+v)dv + M(r+L))\Omega^2\frac{dW}{dx} + M(\omega^2+\Omega^2\sin^2(\theta))W\right)_{x=L} = 0$$

.... (2.37)

$$\left(EI(x)\frac{d^2W}{dx^2} - J\omega^2\frac{dw}{dx}\right)_{x=L} = 0 \qquad \text{.............. (2.38)}$$

2.3 Reduction of the Equation of Motion and the Boundary Conditions to Non-Dimensional Form

The next step is to write the equation of motion and the boundary conditions in non-dimensional form. If the non-dimensional position ξ is defined such that

$$\xi = \frac{x}{L} \qquad \text{.............. (2.39)}$$

and from this definition it follows that:-

$$\frac{d\xi}{dx} = \frac{1}{L} \qquad \text{.............. (2.40)}$$

Also defining the non-dimensional flexural rigidity $f(\xi)$ and the non-dimensional mass per unit length $g(\xi)$ such that :-

$$f(\xi) = \frac{EI(L\xi)}{EI_o} \qquad \text{.............. (2.41)}$$

$$g(\xi) = \frac{\rho A(L\xi)}{\rho A_o} \qquad \text{.............. (2.42)}$$

where EI_0 and ρA_0 may be just numbers, or the flexural rigidity and the mas per unit length at any point or any two different points. But to give them some physical meaning, they are chosen to be the flexural rigidity and mass per unit length at the root (x=0). Taking into account the above defined relations, this leads to:-

$$\frac{d^2}{dx^2}(EI(x)\frac{d^2W}{dx^2}) = \frac{EI_o}{L^4}\frac{d^2}{d\xi^2}(f(x)\frac{d^2W}{d\xi^2}) \qquad \text{............ (2.43)}$$

$$\frac{d}{dx}(\int_x^L \rho A(v)(r+v)dv + M(r+L))\Omega^2\frac{dW}{dx} =$$
$$\rho A_o\frac{d}{d\xi}\left((\int_\xi^1 g(v)(\alpha+v)dv + \frac{M}{\rho A_o L}(\alpha+1))\Omega^2\frac{dW}{d\xi}\right) \qquad \text{............ (2.44)}$$

$$\rho A(x)(\omega^2 + \Omega^2\sin^2(\theta))W = \rho A_o g(\xi)(w^2 - W^2\sin^2(\theta))W \qquad \text{.............. (2.45)}$$

where the root offset ratio is given by $\alpha = r/L$.

Substituting eq's (2.43), (2.44) and (2.45) into eq(2.30), yields:-

$$\frac{EI_o}{L^4}\frac{d^2}{d\xi^2}(f(\xi)\frac{d^2W}{d\xi^2}) - \rho A_o\frac{d}{d\xi}\left((\int_\xi^1 g(v)(\alpha+v)dv + \frac{M}{\rho A_o L}(\alpha+1))\Omega^2\frac{dW}{d\xi}\right) -$$
$$\rho A_o g(\xi)(\omega^2 + \Omega^2\sin^2(\theta))W = 0 \qquad \text{.............. (2.46.a)}$$

dividing eq(2.46.a) by EI_0/L^4, yields :-

$$\frac{d^2}{d\xi^2}(f(\xi)\frac{d^2W}{d\xi^2}) - \frac{\rho A_o L^4\Omega^2}{EI_o}\frac{d}{d\xi}\left((\int_\xi^1 g(v)(\alpha+v)dv + \frac{M}{\rho A_o L}(\alpha+1))\frac{dW}{d\xi}\right) -$$
$$\frac{\rho A_o L^4}{EI_o}(\omega^2 + \Omega^2\sin^2(\theta))g(\xi)W = 0 \qquad \text{............. (2.46)}$$

Now defining the following non-dimensional sets :-

$$\eta = \sqrt{\frac{\rho A_o L^4\Omega^2}{EI_o}} \qquad \text{............. (2.47)}$$

$$\lambda = \left(\frac{\rho A_0 L^4}{EI_0}\right)\left(\omega^2 + \Omega^2\sin^2(\theta)\right) \qquad \text{............. (2.48)}$$

$$\mu = \frac{M}{\rho A_o L} \qquad \text{............. (2.49)}$$

$$\phi = \frac{J}{\rho A_0 L^2} \qquad \text{............ (2.50)}$$

where :-

η: The non-dimensional rotational speed.

λ: The eigenvalues.

μ: The tip mass ratio.

ϕ : The rotational inertia ratio

Using these sets in eq(2.46) leads to :-

$$\frac{d^2}{d\xi^2}(f(\xi)\frac{d^2W}{d\xi^2})-\eta^2\frac{d}{d\xi}\left((\int_{\xi}^{1}g(v)(\alpha+v)dv+\mu(\alpha+1))\frac{dW}{d\xi}\right)-$$

$$\lambda g(\xi)W=0 \qquad \text{..............} \quad (2.51)$$

which is the system equation in non-dimensional form. For the boundary conditions it will be reduced to :-

1: at the root (ξ=0):

a- Fixed root case :-

$$W(0)=0 \qquad \text{..............} \quad (2.52)$$

$$\left.\frac{dW}{d\xi}\right|_{\xi=0}=0 \qquad \text{..............} \quad (2.53)$$

b- For the hinged case :-

$$W(0)=0 \qquad \text{...............}(2.54)$$

$$\left(f(\xi)\frac{d^2W}{d\xi^2}\right)_{\xi=0}=0 \qquad \text{..............} \quad (2.55)$$

c - For the flexible attachment case :-

$$\left(\frac{d}{d\xi}(f(\xi)\frac{d^2W}{d\xi^2})-\eta^2(\int_{\xi}^{1}g(v)(\alpha+v)dv+\mu(\alpha+L))\frac{dW}{d\xi}+\beta W\right)_{x=0}=0$$

$$\text{..............} \quad (2.56)$$

$$\left((f(\xi)\frac{d^2W}{d\xi^2})-\gamma\frac{dW}{d\xi}\right)_{\xi=0}=0 \qquad \text{..............} \quad (2.57)$$

where the translation stiffness ratio (β) and the rotational stiffness ratio (γ) are defined as :-

$$\beta=\frac{K_T L^3}{EI_o}$$

$$\gamma=\frac{K_R L}{EI_o}$$

2- For the tip (ξ=1)

$$\left(\frac{d}{d\xi}\left(f(\xi)\frac{d^2W}{d\xi^2}\right) - \eta^2\left(\int_{\xi}^{1} g(v)(\alpha + v)dv + M(r + L)\right)\frac{dW}{d\xi} + \mu\lambda W\right)_{\xi=1} = 0 \quad \text{.............. (2.58)}$$

$$\left(f(\xi)\frac{d^2W}{d\xi^2} - \phi(\lambda - \eta^2 \sin^2(\theta))\frac{dW}{d\xi}\right)_{\xi=1} = 0 \quad \text{............. (2.59)}$$

For the non-dimensional sets. It is noted that the non-dimensional rotational speed (η) represents parameters related to the cross-section geometry (A_0/I_0), beam material properties (ρ/E), the motion (Ω) in addition to the length of the beam. The tip mass ratio (μ) represents the tip mass. the rotational inertia ratio (ϕ) the rotational inertia of the tip mass. The root offset ratio (α) represents the root offset radius. Both the translation stiffness ratio (β) and the rotational stiffness ratio (γ) represent the translation stiffness and rotational stiffness at the root. In the analysis the rotational inertia ratio will be taken to be equal to the tip mass ratio in numerical value.

Chapter Three

Solution of the Problem of Free Vibration of Rotating Beams with Non-Uniform Cross-Section with General Flexural Rigidity (EI(x)) and Mass Per Unit Length (ρA(x)) Using the Finite Element Method

3.1 Introduction

In this chapter the finite element procedure described in appendix (A) is applied to the problem of free vibration of rotating beams with non-uniform cross-section. Both the flexural rigidity and mass per unit length variations are described by general functions. This method is introduced to just check the result which will be found using the series method.

The problem under consideration was solved using the finite element method for the case where the cross-section properties variations of the beam are given in power series form. A computer program which is presented in Appendix(A) was written to solve this problem. A sample run for an exponentially varying cross-section properties beam is also shown at the end of Appendix (A).

The numerical results obtained from this computer program are compared with the exact results found in Ref[15]. The parameters which are expected to affect the numerical results obtained are studied and presented at the end of this chapter. These parameters are the number of finite elements and the tolerance of the convergence of the computer program solution.

3.2 Solution of the Problem of Free Vibration of Rotating Beams with Non-Uniform Cross-Section Using the Finite Element Method

From chapter two section (2.3) the equation of motion in non-dimensional form for this problem is :-

$$\frac{d^2}{d\xi^2}\left(f(\xi)\frac{d^2W}{d\xi^2}\right)-\eta^2\frac{d}{d\xi}\left(\left(\int_{\xi}^{1}g(v)(\alpha+v)dv+\mu(\alpha+1)\right)\frac{dW}{d\xi}\right)-$$

$$\lambda g(\xi)W=0$$

with the boundary conditions :-

1: at the root (ξ=0):

a- Fixed root case :-

$$W(0)=0$$

$$\left.\frac{dW}{d\xi}\right|_{\xi=0}=0$$

b- For the hinged case :-

$$W(0)=0$$

$$\left(f(\xi)\frac{d^2W}{d\xi^2}\right)_{\xi=0}=0$$

c - For the flexible attachment case :-

$$\left(\frac{d}{d\xi}\left(f(\xi)\frac{d^2W}{d\xi^2}\right)-\eta^2\left(\int_{\xi}^{1}g(v)(\alpha+v)dv+\mu(\alpha+L)\right)\frac{dW}{d\xi}+\beta W\right)_{x=0}=0$$

$$\left((f(\xi)\frac{d^2W}{d\xi^2}) - \gamma\frac{dW}{d\xi}\right)_{\xi=0} = 0$$

2- For the tip (ξ=1)

$$\left(\frac{d}{d\xi}(f(\xi)\frac{d^2W}{d\xi^2}) - \eta^2(\int_{\xi}^{1} g(v)(\alpha+v)dv + M(r+L))\frac{dW}{d\xi} + \mu\lambda W\right)_{\xi=1} = 0$$

$$\left(f(\xi)\frac{d^2W}{d\xi^2} - \mu(\lambda - \eta^2\sin^2(\theta))\frac{dW}{dx}\right)_{\xi=1} = 0$$

Now the finite element method described in Appendix(A) section (A.1) will be applied step by step to solve this deferential equation with its boundary conditions .

1- Discretization of the Domain

The domain of this problem is the beam non-dimensional length from 0 to 1. If the beam is divided into N finite elements of equal lengths. Each element is of length (h) equal to 1/N. For N finite elements there will be (N+1) nodes. The starting node coordinate for element n is :-

$$\xi_n = (n-1)h \qquad \text{..............} \quad (3.1)$$

And the end node coordinate is :

$$\xi_{n+1} = n\,h \qquad \text{..............} \quad (3.2)$$

2- Specifying the Approximating (Interpolation) Function

Since for each element there are four degrees of freedom, the deflection and the slope at the starting and end nodes of the elements. A third order polynomial is chosen to approximate the deflection within any element. Using the non-dimensional coordinate ξ , the interpolation function is :-

$$W(L\xi) = C_1 + C_2\xi + C_3\xi^2 + C_4\xi^3 \qquad \text{..............} \quad (3.3)$$

For any finite element n, the interpolation function should satisfy :-

$$W(L\xi_n) = W_1^n$$

$$\frac{dW(L\xi_n)}{d\xi} = W_2^n$$

$$W(L\xi_{n+1}) = W_3^n$$

$$\frac{dW(L\xi_{n+1})}{d\xi} = W_4^n$$

From these conditions and using eq(3.3), a system of linear algebraic equations is obtained, This system in matrix form is :-

$$\begin{bmatrix} W_1 \\ W_2 \\ W_3 \\ W_4 \end{bmatrix} = \begin{bmatrix} 1 & \xi_n & \xi_n^2 & \xi_n^3 \\ 0 & 1 & 2\xi_n & 3\xi_n^2 \\ 1 & \xi_{n+1} & \xi_{n+1}^2 & \xi_{n+1}^3 \\ 0 & 1 & 2\xi_{n+1} & 3\xi_{n+1}^2 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix}$$

Solving this system and writing W(Lξ) in the form :-

$$W(L\xi) = \sum_{i=1}^{4} N_i(\xi) W_i^n \qquad \text{..............} \quad (3.4)$$

where N_i is the i'th shape function.

The shape functions are given by :-

$$N1(\xi) = 1 - 3\left(\frac{\xi - \xi_n}{\xi_{n+1} - \xi_n}\right)^2 + 2\left(\frac{\xi - \xi_n}{\xi_{n+1} - \xi_n}\right)^3 \qquad \text{..............} \quad (3.5)$$

$$N2(\xi) = (\xi - \xi_n)\left(1 - \frac{\xi - \xi_n}{\xi_{n+1} - \xi_n}\right)^2 \qquad \text{..............} \quad (3.6)$$

$$N3(\xi) = 3\left(\frac{\xi - \xi_n}{\xi_{n+1} - \xi_n}\right)^2 - 2\left(\frac{\xi - \xi_n}{\xi_{n+1} - \xi_n}\right)^3 \qquad \text{.............} \quad (3.7)$$

$$N4(\xi) = (\xi - \xi_n)\left(\left(\frac{\xi - \xi_n}{\xi_{n+1} - \xi_n}\right)^2 - \frac{\xi - \xi_n}{\xi_{n+1} - \xi_n}\right) \qquad \text{..........} \quad (3.8)$$

If the local coordinate $\bar{x}$ is defined such that :-

$$\bar{x} = \xi - \xi_n$$

and from this :-

$$h = \xi_{n+1} - \xi_n$$

The shape functions in terms of the local coordinate $\bar{x}$ are :-

$$N1(\bar{x}) = 1 - 3\left(\frac{\bar{x}}{h}\right)^2 + 2\left(\frac{\bar{x}}{h}\right)^3 \qquad \text{..............} \quad (3.9)$$

$$N2(\bar{x}) = (\bar{x})\left(1 - \frac{\bar{x}}{h}\right)^2 \qquad \text{..............} \quad (3.10)$$

$$N3(\bar{x}) = 3\left(\frac{\bar{x}}{h}\right)^2 - 2\left(\frac{\bar{x}}{h}\right)^3 \qquad \text{..............} \quad (3.11)$$

$$N4(\bar{x}) = (\bar{x})\left(\left(\frac{\bar{x}}{h}\right)^2 - \frac{\bar{x}}{h}\right) \qquad \text{..............} \quad (3.12)$$

these shape functions have the following properties :-

$N1(0) = 1$, $N2(0) = N3(0) = N4(0) = 0$

$N3(h) = 1$, $N1(h) = N2(h) = N4(h) = 0$

The first derivative of the shape functions are :-

$$\frac{dN1(\bar{x})}{d\bar{x}} = \frac{6}{h}\left(\left(\frac{\bar{x}}{h}\right)^2 - \left(\frac{\bar{x}}{h}\right)\right) \qquad \text{..............} \quad (3.13)$$

$$\frac{dN2(\bar{x})}{d\bar{x}} = 1 - 4\left(\frac{\bar{x}}{h}\right) + 3\left(\frac{\bar{x}}{h}\right)^2 \qquad \text{..............} \quad (3.14)$$

$$\frac{dN3(\bar{x})}{d\bar{x}} = \frac{6}{h}\left(\left(\frac{\bar{x}}{h}\right) - \left(\frac{\bar{x}}{h}\right)^2\right) \qquad \text{..............} \quad (3.15)$$

$$\frac{dN4(\bar{x})}{d\bar{x}} = 3\left(\frac{\bar{x}}{h}\right)^2 - 2\left(\frac{\bar{x}}{h}\right) \qquad \text{..............} \quad (3.16)$$

The derivatives of the shape functions have the properties :-

$$\frac{dN2(0)}{d\bar{x}} = 1 \quad , \quad \frac{dN1(0)}{d\bar{x}} = \frac{dN3(0)}{d\bar{x}} = \frac{dN4(0)}{d\bar{x}} = 0$$

$$\frac{dN4(h)}{d\bar{x}} = 1 \quad , \quad \frac{dN1(h)}{d\bar{x}} = \frac{dN2(h)}{d\bar{x}} = \frac{dN3(h)}{d\bar{x}} = 0$$

Also the second derivatives are of the shape functions given by :-

$$\frac{d^2N1(\bar{x})}{d\bar{x}^2} = \frac{6}{h^2}(2\left(\frac{\bar{x}}{h}\right) - 1) \qquad \text{..............} \quad (3.17)$$

$$\frac{d^2N2(\bar{x})}{d\bar{x}^2} = \frac{1}{h}\left(6\left(\frac{\bar{x}}{h}\right) - 4\right) \qquad \text{..............} \quad (3.18)$$

$$\frac{d^2N3(\bar{x})}{d\bar{x}^2} = \frac{6}{h^2}(1 - 2\left(\frac{\bar{x}}{h}\right)) \qquad \text{..............} \quad (3.19)$$

$$\frac{d^2N4(\bar{x})}{d\bar{x}^2} = \frac{1}{h}\left(6\left(\frac{\bar{x}}{h}\right) - 2\right) \qquad \text{..............} \quad (3.20)$$

Both the first and second derivatives of the shape functions are introduced because of their importance in the application of the finite element method and finding the element equation.

3- Development of the Element Equation :

In this step the variation method will be employed to get the element equation. If the system differential equation is multiplied by a trial function $V(\xi)$ and the integration with respect to ξ is taken over the finite element length from ξ_n to ξ_{n+1} .one gets the variation integral as:-

$$\delta = \int_{\xi_n}^{\xi_{n+1}} v\left(\frac{d^2}{d\xi^2}(f(\xi)\frac{d^2W}{d\xi^2}) - \eta^2\frac{d}{d\xi}\left((\int_{\xi}^{1} g(v)(\alpha + v)dv + \mu(\alpha+1))\frac{dW}{d\xi}\right) - \lambda g(\xi)W\right)d\xi \qquad \text{..................} \quad (3.21)$$

integrating the first and second terms by parts, the variation integral equation eq(3.21) becomes:-

$$\delta = \int_{\xi_n}^{\xi_{n+1}} \left(f(\xi)\frac{d^2V}{d\xi^2}\frac{d^2W}{d\xi^2} + \eta^2(\int_{\xi}^{1} g(v)(\alpha+v)dv + \mu(\alpha+1))\frac{dV}{d\xi}\frac{dW}{d\xi} - \right.$$

$$\left. \lambda g(\xi)VW \right)d\xi - \frac{dV}{d\xi}\left(f(\xi)\frac{d^2W}{d\xi^2} \right)\Bigg|_{\xi_n}^{\xi_{n+1}} + V\left[\frac{d}{d\xi}\left[f(\xi)\frac{d^2W}{d\xi^2} \right] - \right.$$

$$\left. \eta^2(\int_{\xi}^{1} g(v)(\alpha+v)dv + \mu(\alpha+1))\frac{dW}{d\xi} \right]\Bigg|_{\xi_n}^{\xi_{n+1}} \quad \ldots\ldots\ldots\ldots (3.22)$$

in terms of the defined local coordinate $\bar{x}$:-

$$\delta = \int_{0}^{h} \left(f(\xi_n+\bar{x})\frac{d^2V}{d\bar{x}^2}\frac{d^2W}{d\bar{x}^2} + \eta^2(\int_{\xi_n+\bar{x}}^{1} g(v)(\alpha+v)dv + \mu(\alpha+1))\frac{dV}{d\bar{x}}\frac{dW}{d\bar{x}} - \right.$$

$$\left. \lambda g(\xi_n+\bar{x})VW \right)d\bar{x} - \frac{dV}{d\bar{x}}\left(f(\xi_n+\bar{x})\frac{d^2W}{d\bar{x}^2} \right)\Bigg|_{0}^{h} + V\left[\frac{d}{d\bar{x}}\left[f(\xi_n+\bar{x})\frac{d^2W}{d\bar{x}^2} \right] - \right.$$

$$\left. \eta^2(\int_{\xi_n+\bar{x}}^{1} g(v)(\alpha+v)dv + \mu(\alpha+1))\frac{dW}{d\bar{x}} \right]\Bigg|_{0}^{h} \quad \ldots\ldots\ldots\ldots\ldots\ldots (3.23)$$

but if defining :-

$$P1 = \left[\frac{d}{d\bar{x}}\left[f(\xi_n+\bar{x})\frac{d^2W}{d\bar{x}^2} \right] - \eta^2(\int_{\xi_n+\bar{x}}^{1} g(v)(\alpha+v)dv + \mu(\alpha+1))\frac{dW}{d\bar{x}} \right]\Bigg|_{0} \quad \ldots\ldots(3.24.a)$$

$$P2 = -\left(f(\xi_n+\bar{x})\frac{d^2W}{d\bar{x}^2} \right)\Bigg|_{0} \quad \ldots\ldots\ldots\ldots\ldots (3.24.b)$$

$$P3 = -\left[\frac{d}{d\bar{x}}\left[f(\xi_n+\bar{x})\frac{d^2W}{d\bar{x}^2} \right] - \eta^2(\int_{\xi_n+\bar{x}}^{1} g(v)(\alpha+v)dv + \mu(\alpha+1))\frac{dW}{d\bar{x}} \right]\Bigg|^{h} \quad \ldots(3.24.c)$$

$$P4 = \left(f(\xi_n+\bar{x})\frac{d^2W}{d\bar{x}^2} \right)\Bigg|^{h} \quad \ldots\ldots\ldots\ldots\ldots\ldots (3.24.d)$$

which are the internal forces on the finite element boundaries then :-

$$\delta = \int_{0}^{h} \left(f(\xi_n+\bar{x})\frac{d^2V}{d\bar{x}^2}\frac{d^2W}{d\bar{x}^2} + \eta^2(\int_{\xi_n+\bar{x}}^{1} g(v)(\alpha+v)dv + \mu(\alpha+1))\frac{dV}{d\bar{x}}\frac{dW}{d\bar{x}} - \right.$$

$$\left. \lambda g(\xi_n+\bar{x})VW \right)d\bar{x} - V(0)P1 - \frac{dV(0)}{d\bar{x}}P2 - V(h)P3 - \frac{dV(h)}{d\bar{x}}P4 \quad \ldots\ldots\ldots (3.25)$$

Setting $V(\bar{x})=W(\bar{x})$ where $W(\bar{x})$ is given by eq(2.57) which is :-

$$W(L\xi) = \sum_{i=1}^{4} N_i(\xi)W_i^n$$

If the functional is minimized with respect to the nodal values, after substituting the interpolation function. This yields :-

$$\int_0^h \left(f(\xi_n+\bar{x})\frac{d^2Ni(\bar{x})}{d\bar{x}^2}\frac{d^2\sum_{j=1}^{4}Nj(\bar{x})}{d\bar{x}^2}+\eta^2\left(\int_{\xi_n+\bar{x}}^{1} g(v)(\alpha+v)dv+\mu(\alpha+1)\right)\frac{dNi(\bar{x})}{d\bar{x}}\frac{d\sum_{j=1}^{4}Nj(\bar{x})}{d\bar{x}} - \lambda g(\xi_n+\bar{x})Ni(\bar{x})\sum_{j=1}^{4}Nj(\bar{x})\right)d\bar{x} = \frac{1}{2}\frac{d}{dWi}\sum_{j=1}^{4}WjPj \qquad \text{.................. (3.26)}$$

where i=1,2,3,4 .

The above equation leads to a system of four linear algebraic equations which can be written in the form :-

$$[[K]-\lambda[M]]^n\{W\}^n=\{f\}^n$$

This equation is the element equation, where [K] and [M] are 4×4 matrices in which :-

$$K_{ij}^n=\int_0^h\left(f(\xi_n+\bar{x})\frac{d^2Ni(\bar{x})}{d\bar{x}^2}\frac{d^2Nj(\bar{x})}{d\bar{x}^2}-\eta^2\left(\int_{\xi_n+\bar{x}}^{1}g(v)(\alpha+v)dv+\mu(\alpha+1)\right)\frac{dNi(\bar{x})}{d\bar{x}}\frac{dNj(\bar{x})}{d\bar{x}}\right)d\bar{x} \qquad \text{............. (3.27)}$$

$$M_{ij}^n=\int_0^h g(\xi_n+\bar{x})Ni(\bar{x})Nj(\bar{x})d\bar{x} \qquad \text{............... (3.28)}$$

where i , j = 1 , 2 , 3 , 4 .

the nodal values vector {W} of the n'th finite element is given by :-

$$\{W\}^n=\begin{Bmatrix}W_1\\W_2\\W_3\\W_4\end{Bmatrix}^n$$

and the force vector is :-

$$\{f\}^n = \frac{1}{2}\left\{\begin{matrix} \frac{d}{dW_1}\sum_{j=1}^{4} WjPj \\ \frac{d}{dW_2}\sum_{j=1}^{4} WjPj \\ \frac{d}{dW_3}\sum_{j=1}^{4} WjPj \\ \frac{d}{dW_4}\sum_{j=1}^{4} WjPj \end{matrix}\right\}^n$$

4- Assembly of the Elements Equation to Formulate the Global System Matrices

If the beam is divided into N finite elements. A (N+1) nodes and 2(N+1) system of linear algebraic equations are obtained. Using the continuity condition of the beam at the nodes, yields that :-

$$W_3^n = W_1^{n+1}$$

$$W_4^n = W_2^{n+1}$$

These relations will be used to fill the system global matrices according to these relations. The relation between the elements of the finite element matrices and the elements of the global system matrices are:-

$$K_{2(n-1)+i,2(n-1)+j} = K_{ij}^n$$

$$M_{2(n-1)+i,2(n-1)+j} = M_{ij}^n$$

where i,j= 1, 2 , 3, 4 .

n : the element number

The force vector for the global system is of size 2(N+1). For the internal nodes n=2,3,4,..., N the corresponding elements of the global force vector {F} are given by :-

$$F_{2(n-1)+1} = \frac{1}{2}\left[\left[\frac{d}{dW_3}\sum_{j=1}^{4} WjPj\right]^{n-1} + \left[\frac{d}{dW_1}\sum_{j=1}^{4} WjPj\right]^{n}\right] = 0$$

$$F_{2(n-1)+2} = \frac{1}{2}\left[\left[\frac{d}{dW_2}\sum_{j=1}^{4} WjPj\right]^{n-1} + \left[\frac{d}{dW_4}\sum_{j=1}^{4} WjPj\right]^{n}\right] = 0$$

They are all zero because they represent the net shear force and bending moment applied at the internal node n. The elements of the global force vector corresponding to the boundary nodes n=1 and n=N+1 depend on the boundary conditions of the beam considered.

5- Imposing of The Boundary conditions

For the boundary nodes n=1 and n=N+1, the boundary conditions were described in chapter two and shown in the beginning of this section. Taking into account the definition of the internal forces on the elements boundaries defined in the element equation derivation, equations (3.24.a), (3.24.b), (3.24.c) and (3.24.d); and the equations of the boundary conditions, the following can be said regarding the boundary conditions:-

1-For the tip at x=L :-

$$F_{2N+1} = \frac{1}{2}\left[\frac{d}{dW_3}\sum_{j=1}^{4} WjPj\right]^N = \frac{1}{2}\left[\frac{d}{dW_3}\left(\mu\lambda W_3^2\right)\right]^N$$

$$= \left(\mu\lambda W_3\right)^N = \mu\lambda W_{2N+1}$$

$$F_{2N+2} = \frac{1}{2}\left[\frac{d}{dW4}\sum_{j=1}^{4} WjPj\right]^N = \frac{1}{2}\left[\frac{d}{dW4}\left(\mu(\lambda - \eta^2 \sin^2\theta) W_4^2\right)\right]^N$$

$$= \left[\left(\mu(\lambda - \eta^2 \sin^2\theta) W_4\right)\right]^N = \mu(\lambda - \eta^2 \sin^2\theta) W_{2N+2}$$

2- For the root. The following cases can be considered :-

a- For fixed root case :-

$$W_1 = 0$$

$$W_2 = 0$$

b- For the hinged root case :-

$$W_1 = 0$$

$$F_2 = \frac{1}{2}\left[\frac{d}{dW2}\sum_{j=1}^{4} WjPj\right]^1 = 0$$

c- For the flexibly attached root-

$$F_1 = \frac{1}{2}\left[\frac{d}{dW_1}\sum_{j=1}^{4} WjPj\right]^1 = \frac{1}{2}\left[\frac{d}{dW_1}\left(-\beta W_1^2\right)\right]^1$$

$$= \left(-\beta W_1\right)^1 = -\beta W_1$$

$$F_2 = \frac{1}{2}\left[\frac{d}{dW_2}\sum_{j=1}^{4} WjPj\right]^1 = \frac{1}{2}\left[\frac{d}{dW_2}\left(-\gamma W_2^2\right)\right]^1$$

$$= \left(-\gamma W_2\right)^1 = -\gamma W_1$$

To apply these conditions if $W_i=0$, the i'th row and column of the global system matrices ([K] and [M]) are removed. If the force vector element $F_i \neq 0$ then it is a function of W_i. For this case F_i is to be subtracted from the element K_{ii} if it doesn't contain λ, if it contains λ, it should be subtracted from the element M_{ii}..

By doing this an eigenvalue problem is gotten. This eigenvalue problem should be solve in order to get the non-dimensional natural frequencies and the mode shapes for the case considered.

6- Solution of the System of Equations and Calculation of the Desired Quantities

Since the system of equations was reduced to an eigenvalue problem after imposing of the boundary conditions the next step is to solve the eigenvalue problem given by :-

$$[K]\{W\} - \lambda[M]\{W\} = \{0\} \qquad \text{..............} \quad (3..29)$$

many numerical methods may be used to solve this eigenvalue problem for the eigenvalues λ and the eigenvectors {W} The best method among them is the matrix deflation method. It gives the eigenvalues λ and the eigenvectors {W} simultaneously. Equation (3.29) can be written in the form:-

$$[K]\{W\} = \lambda[M]\{W\}$$

If this equation is multiplied by $[K]^{-1}$ and divide by λ, one can write:-

$$\frac{1}{\lambda}\{W\} = [K]^{-1}[M]\{W\} = [a]\{W\}$$

If an initial guess for the eigenvector {w} is assumed, say $\{W\}_0$, after substituation in the above equation. One gets:-

$$\frac{1}{\lambda}\{W\}_0 = C_1\{W\}_1$$

where $\{W\}_1$ and C_1 are more accurate approximations for the eigenvector and the reseprocal of eigenvalue. Using the new eigenvector obtained instead of the previous one a more accurate approximation for the eigenvalue and eigenvector will be obtained. This iteration process should be continued until the change in the eigenvector and the change in the eigenvalue are very small, at this moment the eigenvalue and eigenvector found are the recprocal of the eigenvalue and the eigenvector which are required..

For the next eigenvalue and eigenvector, the effect of the last estimated eigenvalue and eigenvector should be elemenated in order to be able to get the next one.This can be done as follows :-

!- First the eigenvector which its effect to be removed should be normalized accourding to the relation :-

$$\{W\}_1^T[M]\{W\}_1 = 1$$

2- The new deflated matrix $[a]_{new}$ can be found using the relation-

$$[a]_{new} = [a]_{old} - \frac{1}{\lambda}\{W\}\{W\}^T[M]$$

after this the new system :-

$$\frac{1}{\lambda}\{W\} = [a]_{new}\{W\}$$

can be solved to get the next eigenvalue and eigenvector as described befor.

The eigenvalues and eigenvectors estimated will be used to get the corresponding natural frequencies and mode shapes for the rotating beam under consideration with a littel farther work.

3.3 Solution of the Problem of Free Vibration of Rotating beams with Non-Uniform Cross-Section for Which the Cross-Section Properties Variation are Given in Series Form by the Finite Element Method

In section (3.2) the finite element method was applied to the general case of rotating beam with non-uniform cross-section. The element equation was found to be :-

$$[[K]-\lambda[M]]^n\{W\}^n=\{f\}^n$$

where:-

$$K_{ij}^n=\int_0^h\left(f(\xi_n+\bar{x})\frac{d^2Ni(\bar{x})}{d\bar{x}^2}\frac{d^2Nj(\bar{x})}{d\bar{x}^2}-\eta^2\left(\int_{\xi_n+\bar{x}}^{1}g(v)(\alpha+v)dv+\mu(\alpha+1)\right)\frac{dNi(\bar{x})}{d\bar{x}}\frac{dNj(\bar{x})}{d\bar{x}}\right)d\bar{x}$$

$$M_{ij}^n=\int_0^h g(\xi_n+\bar{x})Ni(\bar{x})Nj(\bar{x})d\bar{x}$$

where $Ni(\bar{x})$ the i'th shape function which are given by eq(3.16) to eq(320) and i,j = 1, 2, 3, 4.

And the boundary conditions:-

1-For the tip:-

$$F_{2N+1}=\eta^2\mu\lambda W_{2N+1}$$

$$F_{2N+2}=\mu(\lambda-\eta^2\sin^2\theta)W_{2N+2}$$

2- For the root :-

a- For fixed root case :-

$$W_1=0$$

$$W_2 = 0$$

b- For the hinged root case :-

$$W_1 = 0$$

$$F_2 = 0$$

c- For the flexible attached root-

$$F_1 = -\beta W_1$$

$$F_2 = -\gamma W_2$$

If the non-dimensional flexural rigidity $f(\xi)$ and non-dimensional mass per unit length $g(\xi)$ are given by :-

$$f(\xi) = \sum_{k=0}^{m} A_k \xi^k$$
$$g(\xi) = \sum_{k=0}^{m} B_k \xi^k \qquad \text{..............} \quad (3.30)$$

using these equations in the equations which gives the elements of the finite element mass and stiffness matrices yields :-

$$K_{ij}^{n} = \int_{n}^{h} \left(\left(\sum_{k=0}^{m} A_k (\xi_n + \bar{x})^k\right) \frac{d^2 Ni(\bar{x})}{d\bar{x}^2} \frac{d^2 Nj(\bar{x})}{d\bar{x}^2} + \eta^2 \left(\int_{\xi_n + \bar{x}}^{1} \left(\sum_{k=0}^{m} B_k v^k\right)(\alpha + v)\,dv + \mu(\alpha + 1)\right) \frac{dNi(\bar{x})}{d\bar{x}} \frac{dNj(\bar{x})}{d\bar{x}} \right) d\bar{x}$$

$$\text{..............} \quad (3.31)$$

$$M_{ij}^{n} = \int_{0}^{h} \left(\sum_{k=0}^{m} B_k (\xi_n + \bar{x})^k\right) Ni(\bar{x}) Nj(\bar{x})\, d\bar{x} \qquad \text{..............} \quad (3.32)$$

The integrals in eq(3.2) and eq(3.3) can be calculated numerically or analytically. If the polynomials are known analytical integration is preferred. But it is more suitable to use numerical integration in writing computer programs for general purposes.

A computer program shown in Appendix (A) was written in C language to solve for any cross-section properties variation given in power series form of any order. A flow chart for this program is shown in fig(A.1), a complet discribtion of this program is also presented in Appendix (A).

3.4 Numerical Study for the Results Found by the Finite Element Method

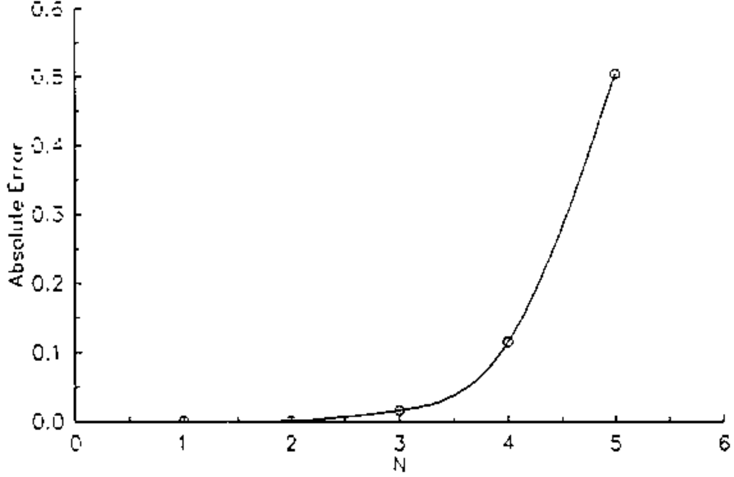
As a first step in the analysis. a comparison of the results obtained from the finite element method computer program with the exact results found in reference [15] is conducted The program was run using ten finite elements and tolerance EPS=1×10^{-6}, for different cases. The results obtained show good agreement with the exact results. The maximum absolute error doesn't exceed 0.6. A lot of results presented in many works in the literature were obtained by the finite element method . But the results obtained from the present described program are much more accurate. This may be due to use double precision variables and using more accurate integration methods in evaluating the element mass and stiffness matrices.

Natural Frequency $(\omega\sqrt{(\rho A_0 L^4)/(EI_0)})$						
Natural Frequ. No.	Exact Values From Ref[15]			Values From The Finite Element Method using 10 Elements		
	$\eta=1$	$\eta=2$	$\eta=3$	$\eta=1$	$\eta=2$	$\eta=3$
1	3.6817	4.1373	4.7973	3.6187	4.1377	4.1377
2	2.1810	2.6149	3.3203	2.1817	2.6156	3.3211
3	1.8148	2.2732	2.9850	1.8577	2.2889	3.0004
4	21.051	21.497	22.236	21.316	21.612	22.350
5	00.012	00.467	01.223	00.516	00.516	01.724

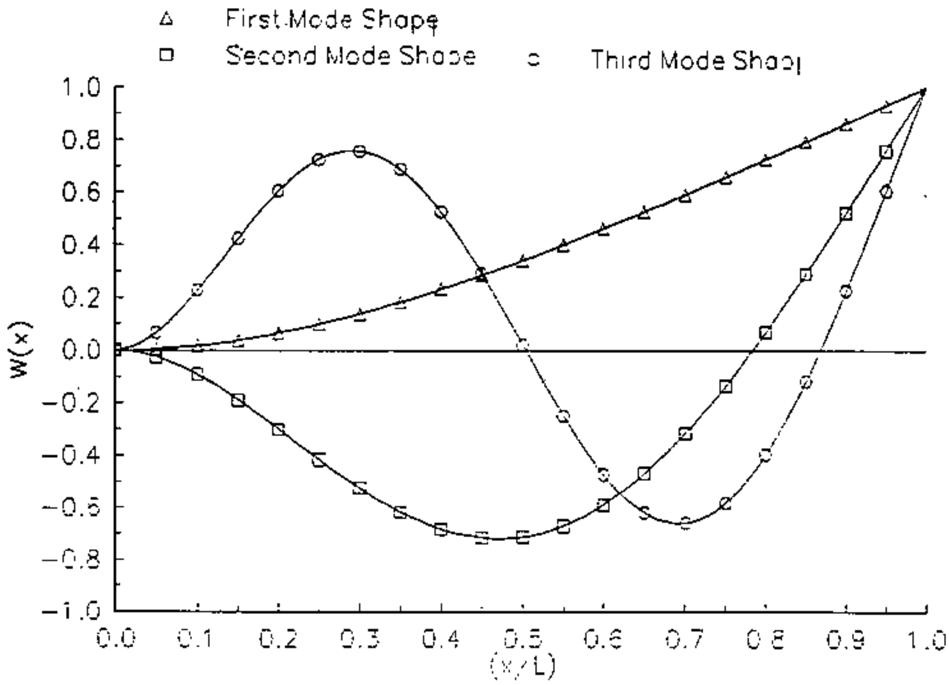
Table(3.1) Comparison of the finite element results with the exact results, for uniform , fixed-root beam with $\mu=0$, $\theta=0$, $\alpha=0$, NFE=10, EPS= 1×10^{-6}

Table (3.1) shows the first five non-dimensional natural frequencies found in reference[15] and those obtained using the finite element program described previously. These results are obtained for the case of uniform cross section beam with zero tip mass ratio, zero setting angle, zero root offset ratio with different values of the non-dimensional rotational speed. The maximum absolute error is about 0.6. It is noted that the maximum absolute error is always in the fifth non-dimensional natural frequency.

Fig(3.2) shows the variation of the absolute error with the non-dimensional natural frequency for non-dimensional rotational speed (η) equal one. For the first non-dimensional natural frequency the absolute error is minimum, it then increases until it reaches its maximum value in the fifth non-dimensional natural frequency. It is noted that the maximum absolute error is associated with the highest non-dimensional natural frequency estimated. This is a general behavior of the solution .



Fig(3.1) Absolute error in finding the first five non-dimensional natural frequencies using the finite element method for uniform, fixed-root beam with η=1.0, μ=0, θ=0,α=0, NFE=10, EPS= 1×10^{-6}



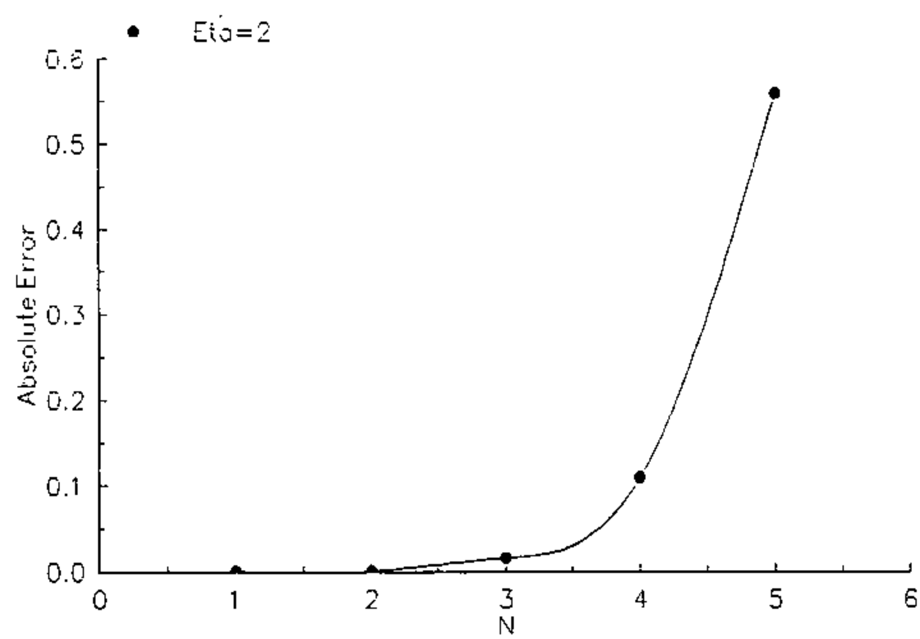
Fig(3.2) The first three mode shapes for uniform, fixed-root beam η=1.0, μ=0, θ=0, α=0, NFE=10, EPS= 1×10^{-6} using the finite element method

Fig(3.3) shows the first three mode shapes for the above considered case with non-dimensional rotational speed equal one.. These mode shape show excellent agreement in their general shapes with that shown in reference [15] for the same case .

For the case of hinged root, uniform cross-section beam with zero tip mass ratio, zero root offset ratio and zero setting angle. Table (3.2) shows the exact values of the non-dimensional natural frequencies found in reference [15] with that obtained from the finite element program. For this case as in the one before a ten finite elements and a tolerance of EPS=1×10^{-6} were used.

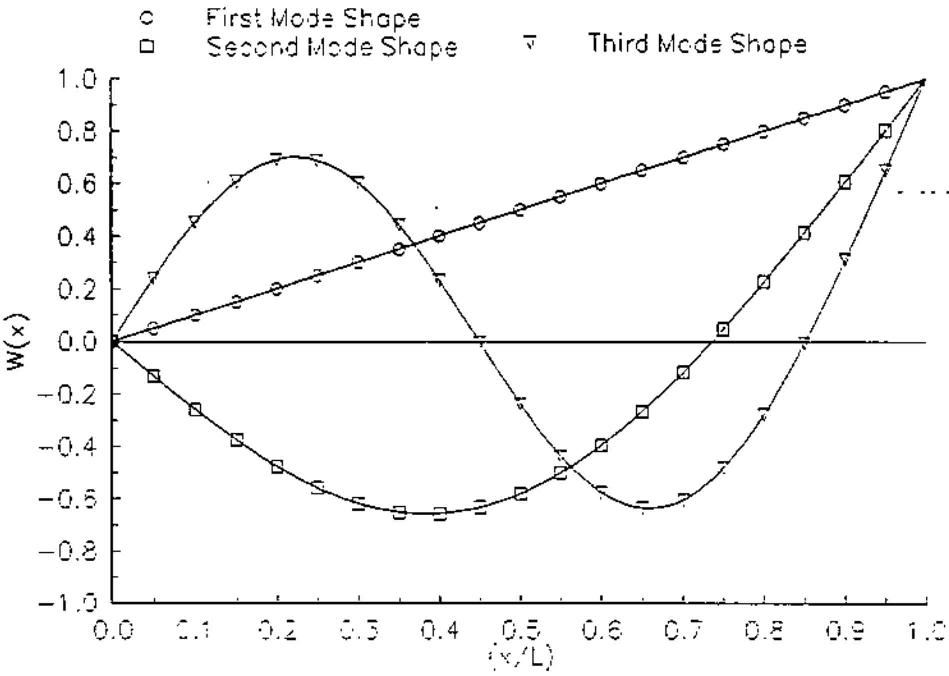
Natural Frequency $(\omega\sqrt{(\rho A_0 L^4)/(EI_0)})$						
Natural Frequ. No.	Exact Values From Ref[15]			Values From The Finite Element Method using 10 Elements		
	η=1	η=2	η=3	η=1	η=2	η=3
1	1.0000	2.000	3.0000	0.9995	2.0005	2.9994
2	15.6242	16.2262	17.1807	15.623.6	16.2266	17.1810
3	50.1437	50.6760	51.5498	50.1511	50.5844	51.5574
4	104.420	104.936	105.789	104.496	105.010	105.862
5	178.440	178.949	179.794	178.900	179.306	180.152

Table(3.2) Comparison of the finite element method results with the exact values, for uniform , hingrd root beam, $\mu=0$, $\theta=0$, $\alpha=0$, NFE=10, EPS= 1×10^{-6}



Fig(3.3) Absolute error in finding the first five non-dimensional natural frequencies using the finite element method for uniform, hinged-root beam $\eta=2.0$, $\mu=0$, $\theta=0,\alpha=0$, NFE=10, EPS= 1×10^{-6}

From these results the finite element solution shows excellent agreement with the exact results. .Fig(3.4) shows the absolute error for this considered case



Fig(3.4) The first three mode shapes for uniform hinged-root beam with η=2.0, μ=0, θ=0, α=0, NFE=10, EPS= 1×10^{-6} using the finite element method

with non-dimensional rotational speed (η) equal two. The maximum error is associated with the highest non-dimensional natural frequency estimated and it is about ,0.5. Fig(3.5) shows the first three mode shapes for this case.

It is observed that for the hinged root cases that the first non-dimensional natural frequency is the non-dimensional rotational speed (η) itself. This is expected because it is known that the first non-dimensional natural frequency is the rigid body mode in which the beam will rotate with angular speed equal (Ω). From the definition of the non-dimensional natural frequency, if one divides and multiply the non-dimensional natural frequency by the rotaional speed (Ω), this yields :-

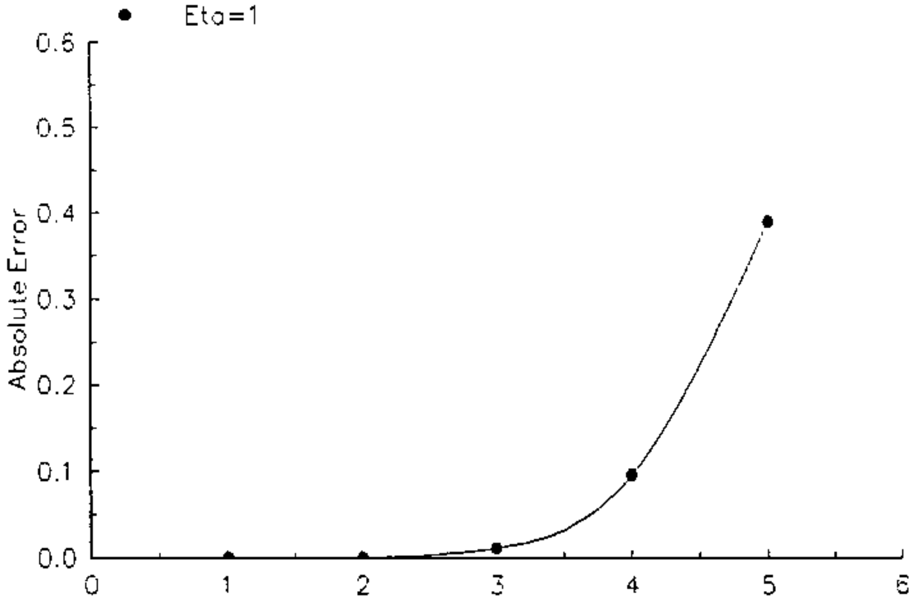
$$\omega\sqrt{\frac{\rho A_o L^4}{EI_o}} = \left(\frac{\omega}{\Omega}\right)\sqrt{\frac{\rho A_o L^4 \Omega^2}{EI_o}} = \eta\left(\frac{\omega}{\Omega}\right)$$

From this equation with the first non-dimensional natural frequencies for the hinged root beam shown in Table(3.2), it is noted that the ratio of the natural frequency to the rotational speed is one. This means that the beam will rotate with angular velocity Ω, This is also clear from the first mode shape in Fig(3.5). This is a general behavior for rotating beams with hinged root.

Natural Frequency $(\omega\sqrt{(\rho A_0 L^4)/(EI_0)})$						
Natural Frequ. No.	Exact Values From Ref[15]			Values From The Finite Element Method using 10 Elements		
	η=1	η=2	η=3	η=1	η=2	η=3
1	5.3903	5.7249	6.2402	5.3902	5.7248	6.2401
2	24.1059	24.4310	24.9149	24.1073	24.4135	24.9158
3	60.0696	60.3669	60.8590	60.0810	50.3783	60.2848
4	113.009	113.307	113.803	113.096	113.394	113.503
5	183.124	183.424	183.923	183.517	183.815	184.425

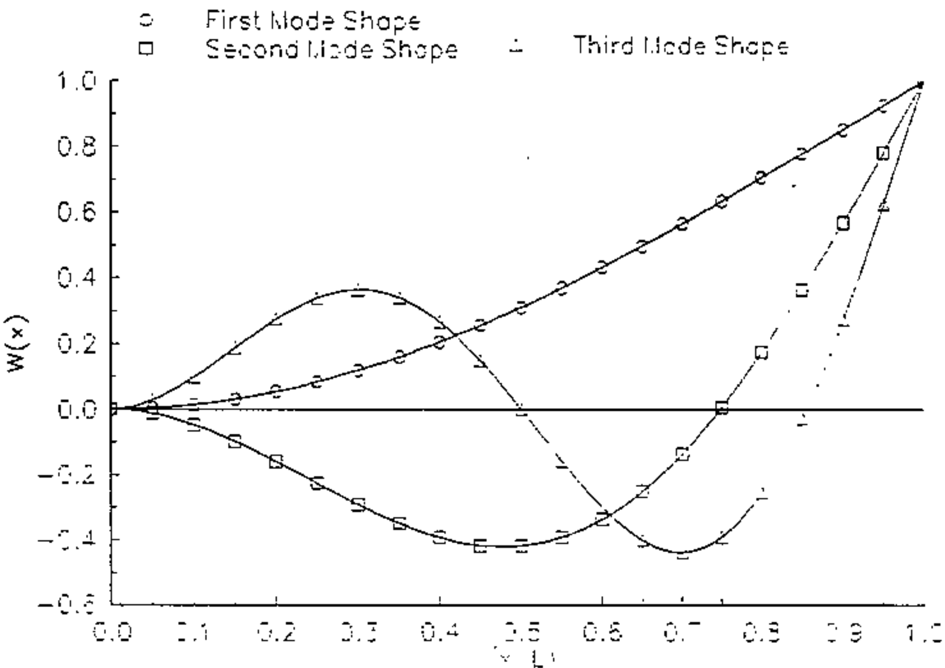
Table(3.3) Comparison of the finite element results with the exact results, for tapered, fixed-root beam with $\mu=0$, $\theta=0$, $\alpha=0$, NFE=10, EPS= 1×10^{-6} for which:-

$$f(\xi) = 1 - 0.95\xi$$
$$g(\xi) = 1 - 0.80\xi$$



Fig(3.5) Absolute error in finding the first five non-dimensional natural frequencies using the finite element method for tapered, fixed-root beam with $\eta=1.0$, $\mu=0$, $\theta=0$, $\alpha=0$, NFE=10, EPS= 1×10^{-6} for which :-

$$f(\xi) = 1 - 0.95\xi$$
$$g(\xi) = 1 - 0.80\xi$$



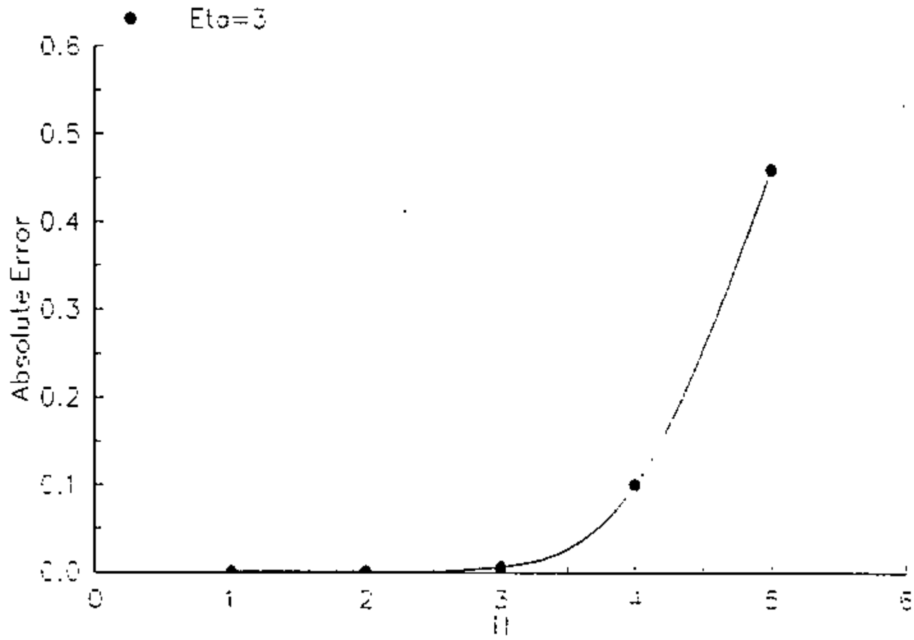
Fig(3.6) The first three shape modes for tapered fixed-root beam with η=1.0, μ=0, θ=0,α=0, NFE=10, EPS= 1×10^{-6} using the finite element method, for which :-

$$f(\xi) = 1 - 0.95\xi$$
$$g(\xi) = 1 - 0.80\xi$$

Natural Frequency $(\omega\sqrt{(\rho A_0 L^4)/(EI_0)})$						
Natural Frequ. No.	Exact Values From Ref[15]			Values From The Finite Element Method using 10 Elements		
	η=1	η=2	η=3	η=1	η=2	η=3
1	1.0000	2.0000	3.0000	0.9992	1.9999	3.0000
2	16.8711	17.2794	17.9388	16.8720	17.2793	17.9393
3	48.5872	48.9395	49.5210	48.5929	48.9453	49.5269
4	97.2823	97.6172	98.1727	97.3493	97.6734	98.2286
5	153.111	163.438	163.983	163.321	163.714	164.259

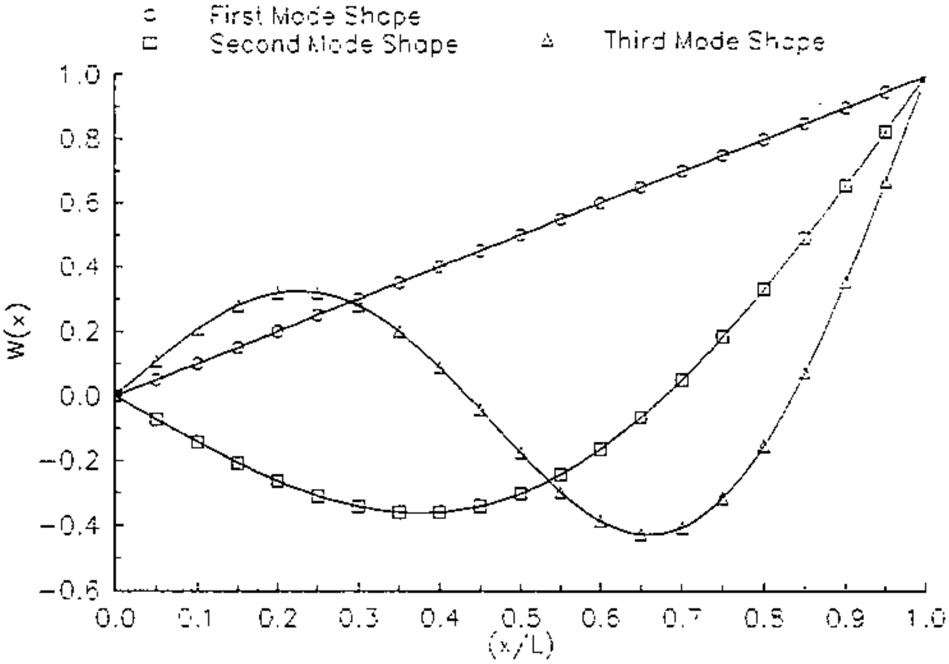
Table(3.4) Comparison of the finite element results with the exact values, for tapered , hinged-root beam with μ=0, θ=0, α=0, NFE=10, EPS= 1×10^{-6} for which :

$$f(\xi) = 1 - 0.95\xi$$
$$g(\xi) = 1 - 0.80\xi$$



Fig(3.7) Absolute error in finding the first five non-dimensional natural frequencies using the finite element method for tapered, hinged-root beam with η=3.0, μ=0, θ=0, α=0, NFE=10, EPS= 1×10^{-6} for which :-

$$f(\xi) = 1 - 0.95\xi$$
$$g(\xi) = 1 - 0.80\xi$$



Fig(3.8) The first three mode shapes for tapered hinged-root beam with with η=3.0, μ=0, θ=0, α=0, NFE=10, EPS= 1×10^{-6} unging the finite element method, for which :-

$$f(\xi) = 1 - 0.95\xi$$
$$g(\xi) = 1 - 0.80\xi$$

In Table(3.3) the results for a fixed root, linearly tapered cross-section beam with zero tip mass, zero setting angle and zero root offset ratio, are shown with the exact results from Ref[15]. These results have excellent agreement with the exact results. Also, this is clear from Fig(3.6) which shows the absolute error for the case with non-dimensional rotational speed (η) equal one. The first three mode shapes for this case are shown in Fig(3.7).

Table(3.4) shows the results for linearly tapered, hinged root beam with zero tip mass ratio, zero setting angle and zero root offset ratio. From the results table the first non-dimensional natural frequency is the rigid body mode as said before. It is always equal to the non-dimensional rotational speed (η) regardless of the cross section variation. The maximum absolute error is in the fifth natural frequency which is expected. Fig(3.8) shows the absolute error against the non-dimensional natural frequency number for the beam under consideration with the non-dimensional rotational speed (η) equal three. Fig(3.9) shows the first three mode shapes for this case .

This program can be used to solve for the special case of non-rotating beam with any cross-section properties variation by just setting the non-dimensional rotational speed (η) to zero. Table(3.5) shows the first five non-dimensional natural frequencies for fixed root, uniform cross-section beam along with the exact non-dimensional natural frequencies.

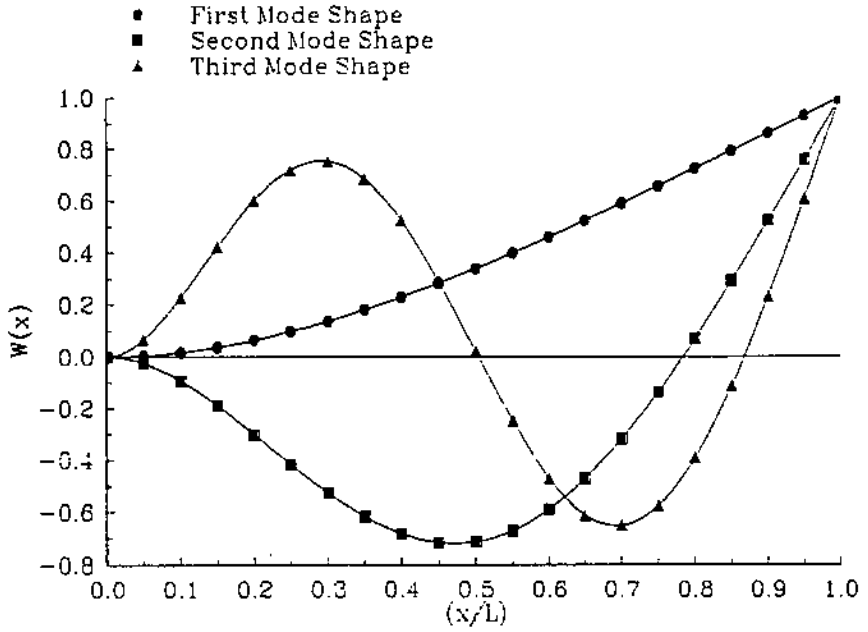
From the results shown in the table. It is noted that the results are very close to the exact results. The maximum absolute error is in the fifth natural frequency and it is about 0.55. The absolute error increase as the number of

Non-Dimensional Natural Frequency $(\omega\sqrt{(\rho A_0 L^4)/(EI_0)})$		
No	Finite Element Method Using ten Elements	Exact Values
1	3.5160	3.5160
2	22.0352	22.0345
3	61.7121	61.6972
4	121.0175	120.902
5	200.3652	199.86

Table(3.5) The first five non-dimensional natural frequencies for uniform non-rotating , fixed-root beam with μ=0, θ=0 and α=0, NFE=10, EPS= 1×10^{-6}

the non-dimensional natural frequency increases. Fig(3.10) shows the first three mode shapes for the case considered above

The running time of the finite element method program was found to be affected by the number of finite elements, the number of non-dimensional natural frequencies required, the cross section properties variation and the value of the tolerance (EPS) in addition to its dependence on the speed of the computer. It was observed .



Fig(3.9) The first three mode shapes for non-rotating, uniform , fixed root beam with μ=0, θ=0, α=0, NFE=10, EPS= 1×10^{-6} using the finite element method

that most of the running time is consumed in finding the global mass and stiffness matrices. The running time in the most simple cases is about a minute and it may reach half an hour or more depending on the case considered.

From the above results and many runs for the program. it was found that the maximum absolute error in the estimated non-dimensional natural frequencies is always associated with the highest non-dimensional natural frequency. It is affected by the order of the beam cross-section properties variation in addition to

1- The number of the finite elements used.

2- The tolerance in the error (EPS).

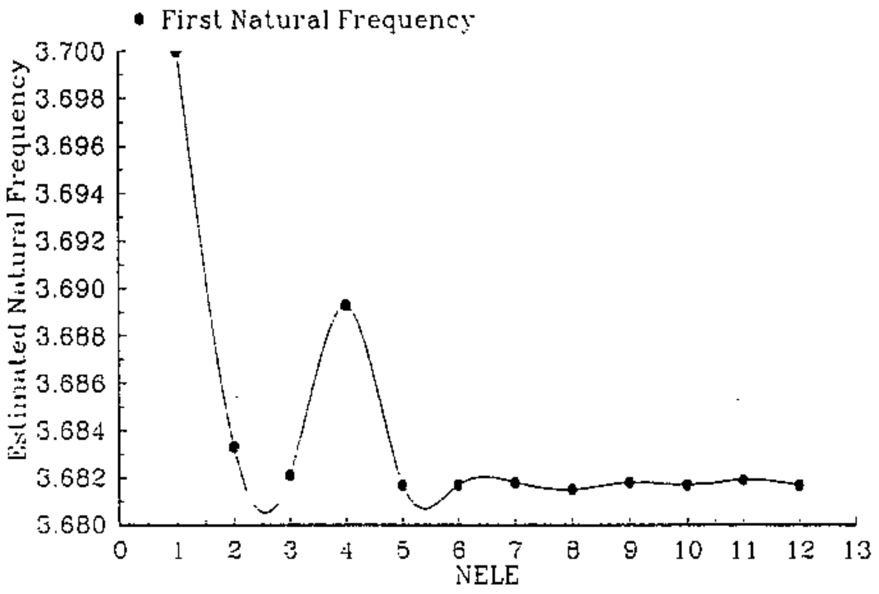
The next two sections presents a numerical study for the effect of each of these factors on the absolute error in the estimated non-dimensional natural frequencies.

3.4.1 The Effect of the Number of Finite Elements (NFELE)

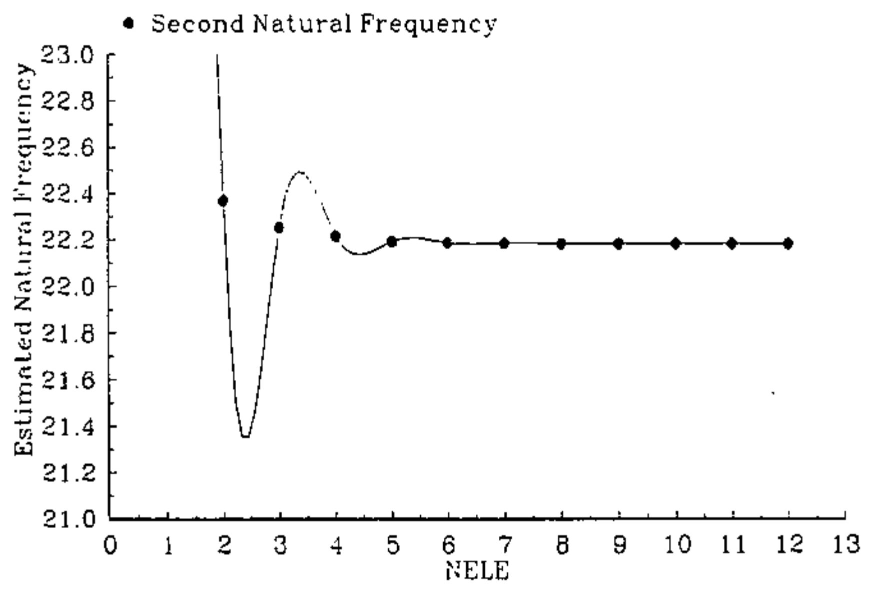
The number of finite elements used highly affects the absolute error presented in the results found for any case considered. Actually the number of finite elements has two important effects. The first one is that the maximum number of natural frequencie that can be found is limited by the number of finite elements (NELE) used in addition to the boundary condition number (BCN). This is because the maximum number of eigenvalues that can be found for (n×n) matrix is n. Since the matrix which is to be solved using (NELE) finite elements is of size 2×(NELE+1)-BCN. From this, it is clear that for one finite element and fixed root beam only two non-dimensional natural frequencies can be found. For two finite elements with fixed root four non-dimensional natural frequencies can be found, and so on.

The second effect of the number of finite elements is related to the absolute error in the results found. Figures (3.11), (3.12), (3.13), (3.14) and (3.15) show the variation of the estimated natural frequency with the number of finite elements for the first five non-dimensional natural frequencies. These results are for a fixed root, uniform cross-section beam with zero tip mass ratio, zero setting angle and zero root offset ratio with non-dimensional rotational speed (η) equal one, using a tolerance EPS=1.0×10^{-6}.

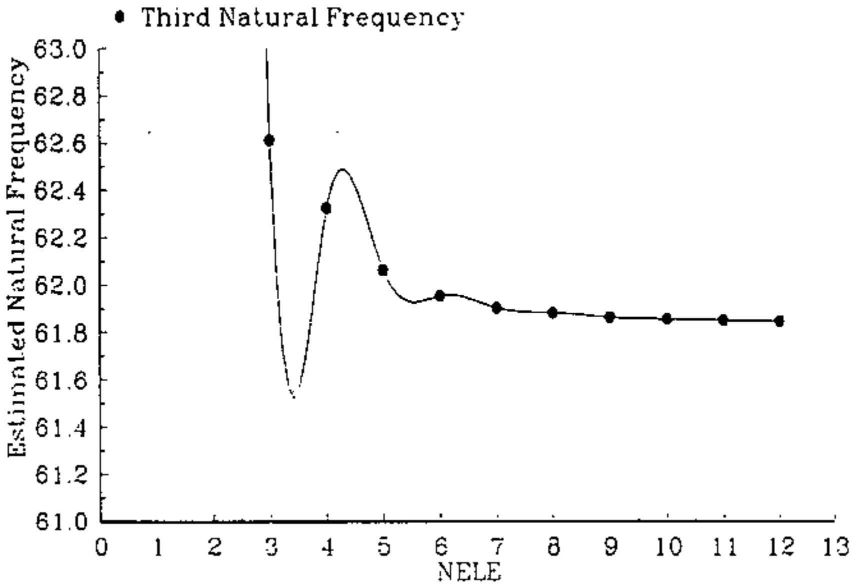
From figures (3.11) and (3.12) as stated above, using one finite element one observes that an approximation only for the first and second non-dimensional natural frequencies can be found. Three finite elements are needed to get a first approximation for the fifth non-dimensional natural frequency. For any non-dimensional natural frequency, until reaching a certain number of finite elements, increasing the number of finite elements improves the approximation. After this number increasing the number of finite elements farther will not improve the approximation of the non-dimensional natural frequency. In other words the natural frequency will be independent of the number of finite elements. This number of finite elements depends on the number of the non-dimensional natural frequency itself. It is clear from the figures that for the first non-dimensional natural frequency five elements are sufficient, for the second natural frequency one needs at least six finite elements to get a good acceptable approximation, But for the fifth natural frequency one needs at least ten finite elements .



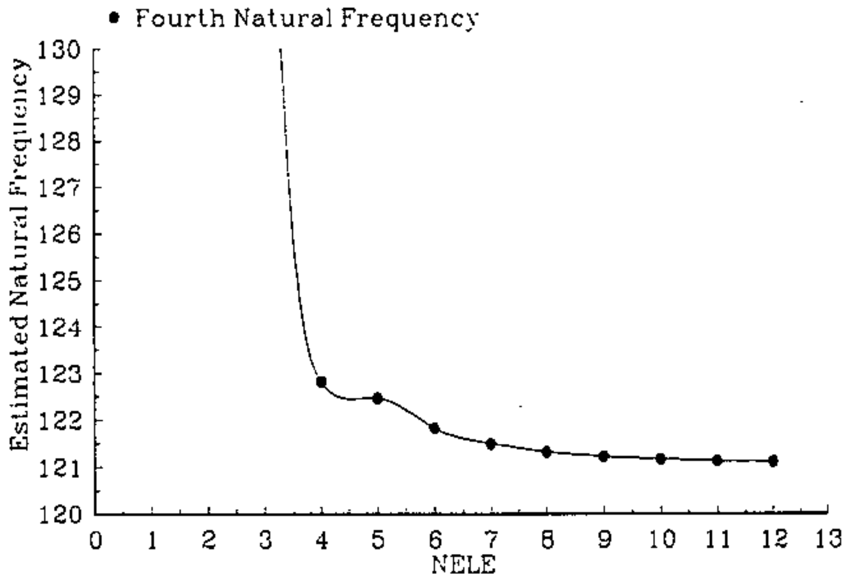
Fig(3.10) The variation of the estimated first non-dimensional natural frequency with the number of finite elements for fixed-root, uniform beam with η=1.0, μ=0, θ=0, α=0, EPS= 1×10^{-6}



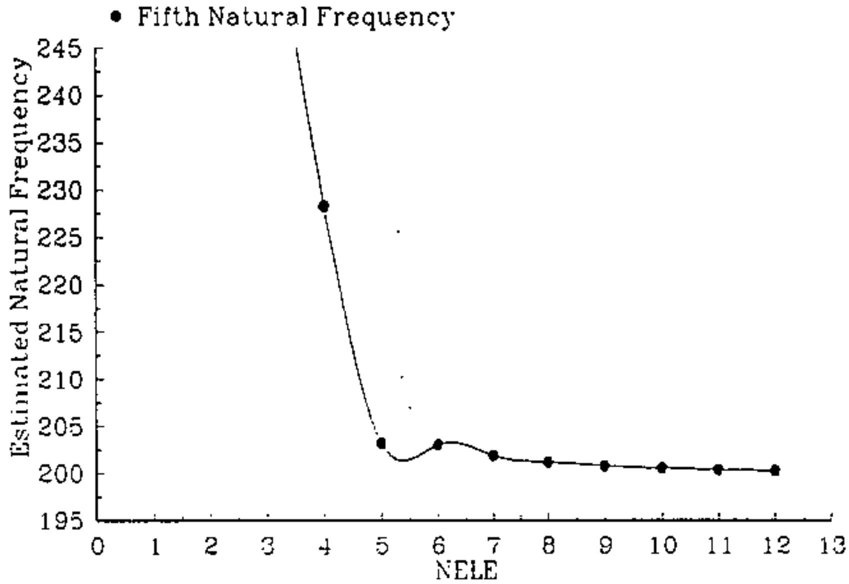
Fig(3.11) The variation of the estimated second non-dimensional natural frequency with the number of finite elements for fixed-root, uniform beam with η=1.0, μ=0, θ=0, α=0, EPS= 1×10^{-6}



Fig(3.12) The variation of the estimated third non-dimensional natural frequency with the number of finite elements for fixed-root, uniform beam with η=1.0, μ=0, θ=0, α=0, EPS= 1×10^{-6}



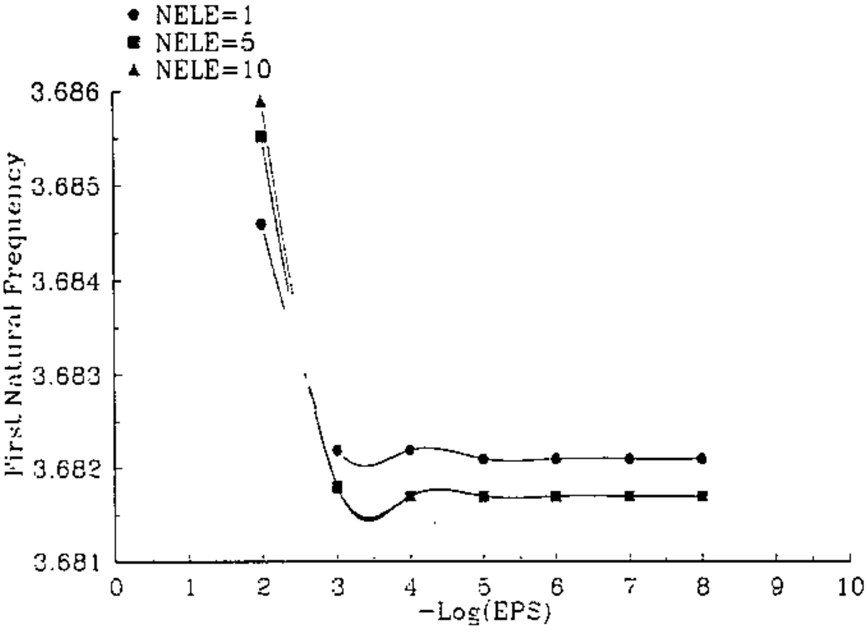
Fig(3.13) The variation of the estimated fourth non-dimensional natural frequency with the number of finite elements for fixed-root, uniform beam with η=1.0, μ=0, θ=0, α=0 EPS= 1×10^{-6}



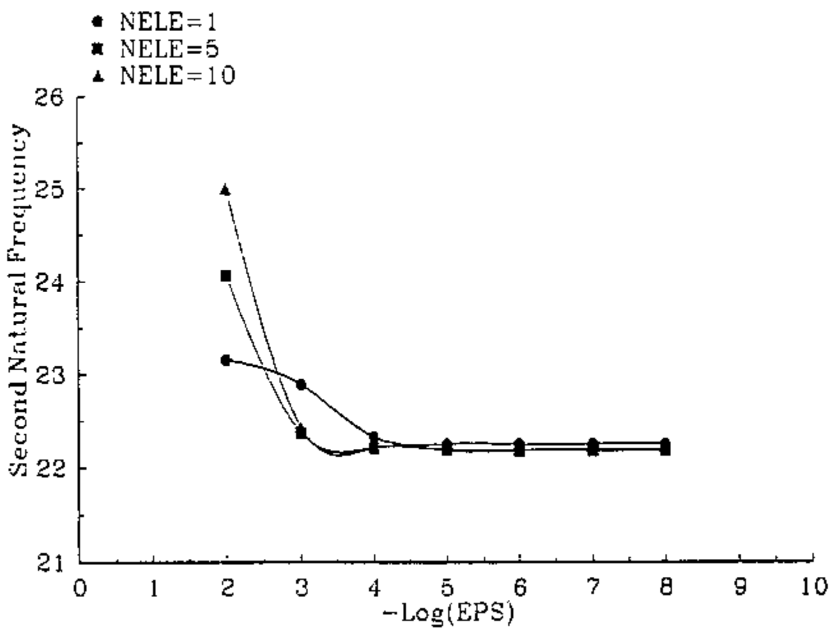
Fig(3.14) The variation of the estimated fifth non-dimensional natural frequency with the number of finite elements for fixed-root, uniform beam with η=1.0, μ=0, θ=0, α=0, EPS= 1×10^{-6}

3.4.2 The Effect of the Tolerance(EPS)

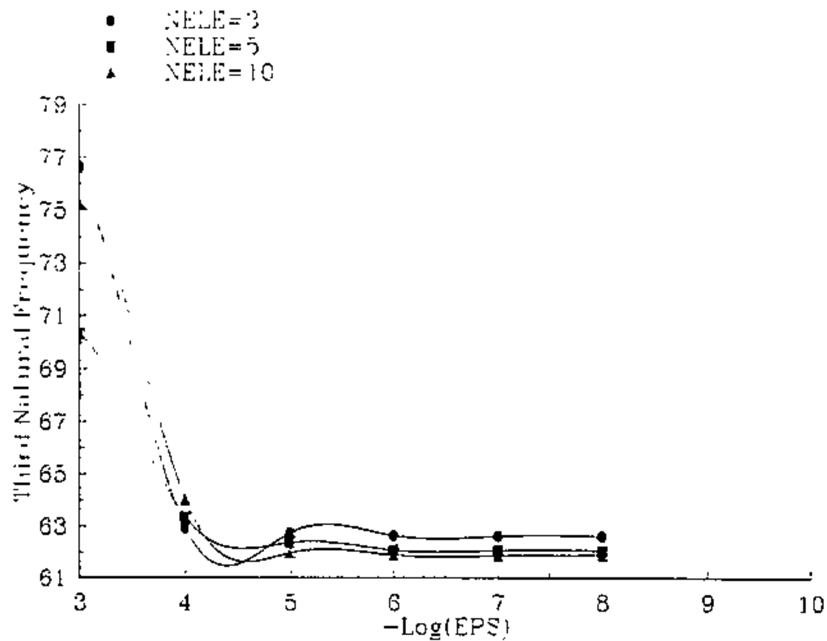
The second parameter which affects the error in the estimated non-dimensional natural frequency is the tolerance. The effect of the tolerance itself is not clear unless it is compounded with the effect of the number of finite elements. Figures (3.16), (3.17), and (3.18) show the variation of the estimated non-dimensional natural frequency with the -Log(EPS), for different numbers of finite element. these results are for the case of fixed root, uniform cross-section beam with zero tip mass, zero setting angle, and zero root offset ratio. From these figures one notes that below a certain value of -Log(EPS), increasing -Log(EPS) will improve the estimated value of the non-dimensional natural frequency, but beyond this value of -Log(EPS) increasing the -Log(EPS) will not improve the estimation. The estimated non-dimensional natural frequency becomes independent of the tolerance (EPS). The estimated non-dimensional natural frequency reaches a steady



Fig(3.15) The variation of the estimated first non-dimensional natural frequency with the tolerance EPS for uniform fixed -root beam with η=1.0, μ=0, θ=0, α=0, NFE=10



Fig(3.16) The variation of the estimated second non-dimensional natural frequency with the tolerance EPS for uniform fixed -root beam with η=1.0, μ=0, θ=0, α=0, NFE=10



Fig(3.17) The variation of the estimated third non-dimensional natural frequency with the tolerance EPS for uniform fixed -root beam with η=1.0, μ=0, θ=0, α=0, NFE=10

value, this steady value has some error which depends on the number of finite elements used. This means that the number of finite elements used is the most important factor affecting the approximation of the non-dimensional natural frequency. From these figures it is also noted that the acceptable value of the tolerance is independent of the non-dimensional natural frequency number. But it should reflect the number of significant figures for which the non-dimensional natural frequency is to be represented by. From the figures it is observed that 10^{-6} is a sufficient value of the tolerance (EPS) to approximate the non-dimensional natural frequencies to four significant figures.

From this study, it was found that if it is required to solve for any rotating beam case using the finite element method program presented, it is suggested first to specify the number of the non-dimensional natural frequencies required. For a certain tolerance (EPS) the problem is solved for different numbers of finite elements and the value of the highest non-dimensional natural frequency required is observed until it becomes independent of the number of finite elements. Using the number of finite elements found the tolerance (EPS) is varied until the highest estimated non-dimensional natural frequencies is independent of the tolerance. Using this procedure, the number of finite elements and the tolerance which will give the best approximation for the solution of the considered case is specified.

Chapter Four

Solution of the Problem of Free Vibration of Rotating Beams with Non-Uniform Cross-Section Using the Power Series Method

4.1 Introduction

In this chapter the problem of free vibration of rotating beams with non-uniform cross section is solved by using the power series method. In section (4.2) the recurrence formula is derived after showing that the solution can be found in a power series form for the case where the cross-section properties variation is in polynomial form or can be written in a polynomial form.

Noting that analytical solution is very difficult to be found for a general beam cross-section properties variation. It is found that computerization is the best way to solve using the series method. A computer program presented in Appendix(B) was written to solve this problem for a general cross-section properties variation. A sample run for exponentially varying cross-section properties bean is also shown in Appendix (B) section (B.3).

In section (4.3) a general study of the numerical results obtained using this program is conducted. In this study the non-dimensional natural frequencies obtained are compared with those found in Ref[15]. The mode shapes obtained are also compared with those found using the finite element method program presented in chapter three. The study is conducted for beams of uniform cross-section and the linearly tapered cross-section properties variations.

Finally, the effect of the approximating function order and the tolerance(EPS) on the estimated non-dimensional natural frequencies are studied. Those two parameters were found to be the most important factors affect the error in the numerical results found.

4.2 Solution by the Power Series Method (The Recurrence Formula)

From section (2.3) the equation of motion of the system in non-dimensional form is :-

$$\frac{d^2}{d\xi^2}\left(f(\xi)\frac{d^2W}{d\xi^2}\right)-\eta^2\frac{d}{d\xi}\left(\left(\int_\xi^1 g(v)(\alpha+v)dv+\mu(\alpha+1)\right)\frac{dW}{d\xi}\right)-$$

$$\lambda g(\xi)W=0$$

With boundary conditions :-

1: at the root (ξ=0):

a- Fixed root case :-

$$W(0)=0$$
$$\left.\frac{dW}{d\xi}\right|_{\xi=0}=0$$

b- For the hinged case :-

$$W(0)=0$$
$$\left(f(\xi)\frac{d^2W}{d\xi^2}\right)_{\xi=0}=0$$

c - For the flexible attachment case :-

$$\left(\frac{d}{d\xi}(f(\xi)\frac{d^2W}{d\xi^2}) - \eta^2(\int_{\xi}^{1} g(v)(\alpha+v)dv + \mu(\alpha+L))\frac{dW}{d\xi} + \beta W\right)_{x=0} = 0$$

$$\left((f(\xi)\frac{d^2W}{d\xi^2}) - \gamma\frac{dW}{d\xi}\right)_{\xi=0} = 0$$

2- For the tip (ξ=1)

$$\left(\frac{d}{d\xi}(f(\xi)\frac{d^2W}{d\xi^2}) - \eta^2(\int_{\xi}^{1} g(v)(\alpha+v)dv + M(r+L))\frac{dW}{d\xi} + \mu\lambda W\right)_{\xi=1} = 0$$

$$\left(f(\xi)\frac{d^2W}{d\xi^2} - \mu(\lambda - \eta^2\sin^2(\theta))\frac{dW}{dx}\right)_{\xi=1} = 0$$

Noting that :-

$$\frac{d}{d\xi}\left(f(\xi)\frac{d^2W}{d\xi^2}\right) = f(\xi)\frac{d^3W}{d\xi^3} + \frac{df(\xi)}{d\xi}\frac{d^2W}{d\xi^2}$$

$$\frac{d^2}{d\xi^2}\left(f(\xi)\frac{d^2W}{d\xi^2}\right) = f(\xi)\frac{d^4W}{d\xi^4} + 2\frac{df(\xi)}{d\xi}\frac{d^3W}{d\xi^3} + \frac{d^2f(\xi)}{d\xi^2}\frac{d^2W}{d\xi^2}$$

.... (4.1)

Also :-

$$\frac{d}{d\xi}\left[\left(\int_{\xi}^{1} g(v)(\alpha+v)dv + \mu(\alpha+1)\right)\frac{dW}{d\xi}\right] = \left(\int_{\xi}^{1} g(v)(\alpha+v)dv + \mu(\alpha+1)\right)\frac{d^2W}{d\xi^2} - g(\xi)(\alpha+\xi)\frac{dW}{d\xi}$$

.... (4.2)

using eq(4.1) and eq(4.2) in the system equation The system equation in its most expanded form is :-

$$f(\xi)\frac{d^4W}{d\xi^4} + 2\frac{df(\xi)}{d\xi}\frac{d^3W}{d\xi^3} + \frac{d^2f(\xi)}{d\xi^2}\frac{d^2W}{d\xi^2} - \eta^2\left(\int_{\xi}^{1} g(v)(\alpha+v)dv + \mu(\alpha+1)\right)\frac{d^2W}{d\xi^2} + \eta^2 g(\xi)(\alpha+\xi)\frac{dW}{d\xi} - \lambda g(\xi)W = 0$$

.... (4.3)

From equation (4.3) it is noted that the system equation is a fourth order linear differential equation with variable coefficients. It is expected that the solution will have four arbitrary constants. Moreover the point $\xi=0$ is expected to be an ordinary point since **f(0)≠0**. This because the point $\xi=0$ is the root of the beam which has either fixed, hinged or flexible attachment. This means that both the cross-section area and the flexural rigidity of the beam can never be zero ..

For the interval $0\le\xi\le1$ the second moment of area can not be zero anyhow .If the non-dimensional flexural rigidity is zero (**f(ξ)=0**) within the beam span, this means that the cross-section area is also zero. Physically this means that the end of the beam is reached ..

For the tip of the beam if the case is such that **f(1)=0**. Also this implies that **g(1)=0**. From this the beam will not have a tip mass. logically this implies that both the shear force and bending moment are zero. For this case the boundary conditions at the tip where $\xi=1$ will be reduced to :-

$$\left(\frac{d^2W}{d\xi^2}\right)_{\xi=1} = 0$$

$$\left(\frac{d^3W}{d\xi^3}\right)_{\xi=1} = 0$$

and no solution may be found around the point $\xi=1$. but since this case is very rare in practical problems it can be neglected.

From all of this as stated in appendix B. The series solution is expected to be of the form :-

$$W(L\xi) = \sum_{n=0}^{\infty} C_n \xi^n \qquad \text{.............. (4.4)}$$

As stated before, the flexural rigidity and mass per unit length are in polynomial form or can be expanded as a power series form by using the Taylor series expansion. The cross-section properties variations are assumed to be given by :-

$$f(\xi)=\sum_{k=0}^{\infty}A_k\xi^k$$
$$g(\xi)=\sum_{k=0}^{\infty}B_k\xi^k \qquad \text{.............. (4.5)}$$

From eq(4.4) the following can be written:-

$$\frac{dW}{d\xi}=\sum_{n=1}^{\infty}nC_n\xi^{n-1} \qquad \text{.............. (4.6.a)}$$
$$\frac{d^2W}{d\xi^2}=\sum_{n=2}^{\infty}n(n-1)C_n\xi^{n-2} \qquad \text{.............. (4.6.b)}$$
$$\frac{d^3W}{d\xi^3}=\sum_{n=3}^{\infty}n(n-1)(n-2)C_n\xi^{n-3} \qquad \text{.............. (4.6.c)}$$
$$\frac{d^4W}{d\xi^4}=\sum_{n=4}^{\infty}n(n-1)(n-2)(n-3)C_n\xi^{n-4} \qquad \text{.............. (4.6.d)}$$

Also from the cross-section properties variations eq(4.5), one can write :-

$$\frac{df(\xi)}{d\xi}=\sum_{k=1}^{m}kA_k\xi^{k-1}$$
$$\frac{d^2f(\xi)}{d\xi^2}=\sum_{k=2}^{m}k(k-1)A_k\xi^{k-2} \qquad \text{.............. (4.7.a)}$$

Using the non-dimensional mass variation from eq(4.5), one writes :-

$$\int_{\xi}^{1}g(v)(\alpha+v)dv=\int_{\xi}^{1}\left(\sum_{k=0}^{m}B_kv^k\right)(\alpha+v)dv=\int_{\xi}^{1}\left(\sum_{k=0}^{m}\alpha B_kv^k\right)dv+\int_{\xi}^{1}\left(\sum_{k=0}^{m}B_kv^{k+1}\right)dv$$
$$=\sum_{k=0}^{m}B_k\left(\frac{\alpha}{k+1}+\frac{1}{k+2}\right)-\sum_{k=0}^{m}\frac{\alpha B_k}{k+1}\xi^{k+1}-\sum_{k=0}^{m}\frac{B_k}{k+2}\xi^{k+2}$$

$$\text{.............. (4.8.a)}$$

$$g(\xi)(\alpha+\xi)=\left(\sum_{k=0}^{m}B_k\xi^k\right)(\alpha+\xi)=\sum_{k=0}^{m}\alpha B_k\xi^k+\sum_{k=0}^{m}B_k\xi^{k+1} \qquad \text{.............. (4.8.b)}$$

To get the recurrence formula From eq(4.3) taking term by term and simplifying,. For the first term :-

$$f(\xi)\frac{d^4W}{d\xi^4}=\left(\sum_{k=0}^{m}A_k\xi^k\right)\left(\sum_{n=4}^{\infty}n(n-1)(n-2)(n-3)C_n\xi^{n-4}\right)$$
$$=\sum_{k=0}^{m}\sum_{n=4}^{\infty}n(n-1)(n-2)(n-3)A_kC_n\xi^{n+k-4}$$

by shifting the index n, yields :-

$$f(\xi)\frac{d^4W}{d\xi^4}=\sum_{k=0}^{m}\sum_{n=k}^{\infty}(n-k+4)(n-k+3)(n-k+2)(n-k+1)A_kC_{n-k+4}\xi^n$$

and after interchanging the summation, The final form of the first term in eq(4.3) is :-

$$f(\xi)\frac{d^4W}{d\xi^4} = \sum_{n=0}^{\infty}\sum_{k=0}^{q1}(n-k+4)(n-k+3)(n-k+2)(n-k+1)A_k C_{n-k+4}\xi^n$$

.... (4.9.a)

where q1=m if n>m

q1=n if n≤m

For the second term in eq(4.3) :-

$$\frac{df(\xi)}{d\xi}\frac{d^3W}{d\xi^3} = \left(\sum_{k=1}^{m} kA_k\xi^{k-1}\right)\left(\sum_{n=3}^{\infty} n(n-1)(n-2)C_n\xi^{n-3}\right)$$

$$= \sum_{k=1}^{m}\sum_{n=3}^{\infty} k\,n(n-1)(n-2)A_k C_n\xi^{n+k-4}$$

shifting the index n :-

$$\frac{df(\xi)}{d\xi}\frac{d^3W}{d\xi^3} = \sum_{k=1}^{m}\sum_{n=3}^{\infty} k\,(n-k+4)(n-k+3)(n-k+2)A_k C_{n-k+4}\xi^n$$

after interchanging the summation, and by adding some terms which are zeroes, the above equation is reduced to :-

$$\frac{df(\xi)}{d\xi}\frac{d^3W}{d\xi^3} = \sum_{n=0}^{\infty}\sum_{k=1}^{q1} k\,(n-k+4)(n-k+3)(n-k+2)A_k C_{n-k+4}\xi^n$$

.... (4.9.b)

where q1 is as defined above

For the third term in eq(4.3) :-

$$\frac{d^2f(\xi)}{d\xi^2}\frac{d^2W}{d\xi^2} = \left(\sum_{k=2}^{m} k(k-1)A_k\xi^{k-2}\right)\left(\sum_{n=2}^{\infty} n(n-1)C_n\xi^{n-2}\right)$$

$$= \sum_{k=2}^{m}\sum_{n=2}^{\infty} k(k-1)n(n-1)A_k C_n\xi^{n-k+4}$$

by shifting the index n :-

$$\frac{d^2f(\xi)}{d\xi^2}\frac{d^2W}{d\xi^2} = \sum_{k=2}^{m}\sum_{n=k-2}^{\infty} k(k-1)(n-k+4)(n-k+3)A_k C_{n-k+4}\xi^n$$

after changing the order of the summation, and adding some zero terms :-

$$\frac{d^2f(\xi)}{d\xi^2}\frac{d^2W}{d\xi^2} = \sum_{n=2}^{\infty}\sum_{k=2}^{q1} k(k-1)(n-k+4)(n-k+3)A_k C_{n-k+4}\xi^n$$

.... (4.9.c)

where q1 is also as defined above.

Using eq(4.9.a), eq(4.9.b) and eq(4.9.c). eq(4.1) is reduced to :-

$$\frac{d^2}{d\xi^2}\left(f(\xi)\frac{d^2W}{d\xi^2}\right)=\sum_{n=0}^{\infty}\sum_{k=1}^{q1}(n-k+4)(n-k+3)(n-k+2)(n-k+1)$$

$$A_kC_{n-k+4}\xi^n+2\sum_{n=0}^{\infty}\sum_{k=1}^{q1}k(n-k+4)(n-k+3)(n-k+2)A_kC_{n-k+4}\xi^n+$$

$$\sum_{n=2}^{\infty}\sum_{k=2}^{q1}k(k-1)(n-k+4)(n-k+3)A_kC_{n-k+4}\xi^n$$

by rearranging and noting that all of the series can be started from k=0, yields

$$\frac{d^2}{d\xi^2}\left(f(\xi)\frac{d^2W}{d\xi^2}\right)=\sum_{n=0}^{\infty}\sum_{k=0}^{q1}(n-k+4)(n-k+3)A_kC_{n-k+4}\Big((n-k+2)(n-k+1)+$$

$$2k(n-k+2)+k(k-1)\Big)\xi^n$$

simplifying :-

$$\frac{d^2}{d\xi^2}\left(f(\xi)\frac{d^2W}{d\xi^2}\right)=\sum_{n=0}^{\infty}\sum_{k=1}^{q1}(n-k+4)(n-k+3)(n+2)(n+1)A_kC_{n-k+4}\xi^n$$

.... (4.10)

where q1=m if n>m

q1=n if n≤m

From eq(4.3) and using eq(4.8.a) the fourth term becomes :-

$$\left(\int_{\xi}^{1}g(v)(\alpha+v)dv+\mu(\alpha+1)\right)\frac{d^2W}{d\xi^2}=\left(\sum_{k=0}^{m}B_k\left(\frac{\alpha}{k+1}+\frac{1}{k+2}\right)+\mu(\alpha+1)-\sum_{k=0}^{m}\frac{\alpha B_k}{k+1}\xi^{k+1}-\right.$$ ....

$$\left.\sum_{k=0}^{m}\frac{B_k}{k+2}\xi^{k+2}\right)\left(\sum_{n=2}^{\infty}n(n-1)C_n\xi^{n-2}\right)$$

.... (4.11)

if term by term from this equation, eq(4.11) is taken. For the first term :-

$$\left(\sum_{k=0}^{m}B_k\left(\frac{\alpha}{k+1}+\frac{1}{k+2}\right)+\mu(\alpha+1)\right)\left(\sum_{n=2}^{\infty}n(n-1)C_n\xi^{n-2}\right)=$$

$$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{m}B_k\left(\frac{\alpha}{k+1}+\frac{1}{k+2}\right)\right)(n+2)(n+1)C_{n+2}\xi^n$$

.... (4.12.a)

The second term :-

$$\left(\sum_{k=0}^{m}\frac{\alpha B_k}{k+1}\xi^{k+1}\right)\left(\sum_{n=2}^{\infty}n(n-1)C_n\xi^{n-2}\right)=\sum_{k=0}^{m}\sum_{n=2}^{\infty}\frac{\alpha n(n-1)}{k+1}B_kC_n\xi^{n+k-1}$$

$$=\sum_{k=0}^{m}\sum_{n=k+1}^{\infty}\frac{\alpha(n-k+1)(n-k)}{k+1}B_kC_{n-k+1}\xi^{n}$$

$$=\sum_{n=0}^{\infty}\sum_{k=0}^{q2}\frac{\alpha(n-k+1)(n-k)}{k+1}B_kC_{n-k+1}\xi^{n}$$

.... (4.12.b)

where :- $q2=n$ if $n<m$

$q2=m$ if $n\geq m$

for the last term :-

$$\left(\sum_{k=0}^{m}\frac{B_k}{k+2}\xi^{k+2}\right)\left(\sum_{n=2}^{\infty}n(n-1)C_n\xi^{n-2}\right)=\sum_{k=0}^{m}\sum_{n=2}^{\infty}\frac{n(n-1)}{k+2}B_kC_n\xi^{n+k}$$

$$=\sum_{k=0}^{m}\sum_{n=k+2}^{\infty}\frac{(n-k)(n-k-1)}{k+2}B_kC_{n-k}\xi^{n}$$

$$=\sum_{n=0}^{\infty}\sum_{k=0}^{q3}\frac{(n-k)(n-k-1)}{k+2}B_kC_{n-k}\xi^{n}$$

.... (4.12.c)

Using equations (4.12.a), (4.12.b) and (4.12.c) in equation (4.11). After collecting the terms and rearranging one gets :-

$$\left(\int_{\xi}^{1}g(v)(\alpha+v)dv+\mu(\alpha+1)\right)\frac{d^2W}{d\xi^2}=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{m}B_k\left(\frac{\alpha}{k+1}+\frac{1}{k+2}\right)+\mu(\alpha+1)\right)(n+2)(n+1)C_{n+2}\xi^{n}-$$

$$\sum_{n=0}^{\infty}\sum_{k=0}^{q2}\frac{\alpha(n-k+1)(n-k)}{k+1}B_kC_{n-k+1}\xi^{n}-\sum_{n=0}^{\infty}\sum_{k=0}^{q3}\frac{(n-k)(n-k-1)}{k+2}B_kC_{n-k}\xi^{n}$$

.... (4.13)

where q2 is as defined above.

For the fifth term in eq(4.3). and from eq(4.8.b):-

$$g(\xi)(\alpha+\xi)\frac{dW}{d\xi}=\left(\sum_{k=0}^{m}B_k\xi^{k}\right)(\alpha+\xi)\left(\sum_{n=1}^{\infty}nC_n\xi^{n-1}\right)$$

$$=\left(\sum_{k=0}^{m}\alpha B_k\xi^{k}+\sum_{k=0}^{m}B_k\xi^{k+1}\right)\left(\sum_{n=1}^{\infty}nC_n\xi^{n-1}\right)$$

.... (4.14)

Taking term by term from eq(4.14). For the first term :-

$$\left(\sum_{k=0}^{m}\alpha B_k\xi^k\right)\left(\sum_{n=1}^{\infty}nC_n\xi^{n-1}\right)=\sum_{k=0}^{m}\sum_{n=1}^{\infty}\alpha nB_kC_n\xi^{n+k-1}$$

$$=\sum_{k=0}^{m}\sum_{n=k}^{\infty}\alpha(n-k+1)B_kC_{n-k+1}\xi^n$$

$$=\sum_{n=0}^{\infty}\sum_{k=0}^{q2}\alpha(n-k+1)B_kC_{n-k+1}\xi^n$$

.... (4.15.a)

where q2 as defined above

For the second term in eq(4.14):-

$$\left(\sum_{k=0}^{m}B_k\xi^{k+1}\right)\left(\sum_{n=1}^{\infty}nC_n\xi^{n-1}\right)=\sum_{k=0}^{m}\sum_{n=1}^{\infty}nB_kC_n\xi^{n+k}$$

$$=\sum_{k=0}^{m}\sum_{n=k+1}^{\infty}(n-k)B_kC_{n-k}\xi^n$$

$$=\sum_{n=0}^{\infty}\sum_{k=0}^{q2}(n-k)B_kC_{n-k}\xi^n$$

.... (4.15.b)

where q2 as defined above. Substituting equations (4.15.a) and (4.15.b) in eq(4.14), and after simplifying, one can write :-

$$g(\xi)(\alpha+\xi)\frac{dW}{d\xi}=\sum_{n=0}^{\infty}\sum_{k=0}^{q2}\alpha(n-k+1)B_kC_{n-k+1}\xi^n+\sum_{n=0}^{\infty}\sum_{k=0}^{q2}(n-k)B_kC_{n-k}\xi^n$$

.... (4.16)

For the last term in eq(4.3) :-

$$\lambda g(\xi)W=\lambda\left(\sum_{k=0}^{\infty}B_k\xi^k\right)\left(\sum_{n=0}^{\infty}C_n\xi^n\right)$$

$$=\sum_{k=0}^{m}\sum_{n=0}^{\infty}\lambda B_kC_n\xi^{n+k}$$

$$=\sum_{k=0}^{m}\sum_{n=k}^{\infty}\lambda B_kC_{n-k}\xi^n$$

$$=\sum_{n=0}^{\infty}\sum_{k=0}^{q2}\lambda B_kC_{n-k}\xi^n$$

.... (4.17)

where q2 as defined previously.

From equations (4.13), (4.16), (4.17) , one can write:-

$$-\eta^2\left(\int_\xi^1 g(v)(\alpha+v)dv+\mu(\alpha+1)\right)\frac{d^2W}{d\xi^2}+\eta^2 g(\xi)(\alpha+\xi)\frac{dW}{d\xi}-\lambda g(\xi)W=$$

$$-\eta^2\sum_{n=0}^{\infty}\left(\sum_{k=0}^{m}B_k\left(\frac{\alpha}{k+1}+\frac{1}{k+2}\right)+\mu(\alpha+1)\right)(n+2)(n+1)C_{n+2}\xi^n+$$

$$\sum_{n=0}^{\infty}\sum_{k=0}^{q2}\frac{\alpha(n-k+1)(n-k)}{k+1}\eta^2 B_k C_{n-k+1}\xi^n+\sum_{n=0}^{\infty}\sum_{k=0}^{q3}\frac{(n-k)(n-k-1)}{k+2}\eta^2 B_k C_{n-k}\xi^n+$$

$$\sum_{n=0}^{\infty}\sum_{k=0}^{q2}\alpha\eta^2(n-k+1)B_k C_{n-k+1}\xi^n+\sum_{n=0}^{\infty}\sum_{k=0}^{q2}(n-k)\eta^2 B_k C_{n-k}\xi^n-\sum_{n=0}^{\infty}\sum_{k=0}^{q2}\lambda B_k C_{n-k}\xi^n$$

By collecting the terms this leads to :-

$$-\eta^2\left(\int_\xi^1 g(v)(\alpha+v)dv+\mu(\alpha+1)\right)\frac{d^2W}{d\xi^2}+\eta^2 g(\xi)(\alpha+\xi)\frac{dW}{d\xi}-\lambda g(\xi)W=$$

$$-\sum_{n=0}^{\infty}\left(\sum_{k=0}^{m}B_k\left(\frac{\alpha}{k+1}+\frac{1}{k+2}\right)+\mu(\alpha+1)\right)\eta^2(n+2)(n+1)C_{n+2}\xi^n-$$

$$\sum_{n=0}^{\infty}\sum_{k=0}^{q2}\left(\frac{(n-k)}{k+1}+1\right)\alpha\eta^2(n-k+1)B_k C_{n-k+1}\xi^n+$$

$$\sum_{n=0}^{\infty}\sum_{k=0}^{q2}\left(\frac{(n-k-1)}{k+2}+1\right)\eta^2(n-k)B_k C_{n-k}\xi^n-\sum_{n=0}^{\infty}\sum_{k=0}^{q2}\lambda B_k C_{n-k}\xi^n$$

simplifying this equation, one gets :-

$$-\eta^2\left(\int_\xi^1 g(v)(\alpha+v)dv+\mu(\alpha+1)\right)\frac{d^2W}{d\xi^2}+\eta^2 g(\xi)(\alpha+\xi)\frac{dW}{d\xi}-\lambda g(\xi)W=$$

$$-\sum_{n=0}^{\infty}\left(\sum_{k=0}^{m}B_k\left(\frac{\alpha}{k+1}+\frac{1}{k+2}\right)+\mu(\alpha+1)\right)\eta^2(n+2)(n+1)C_{n+2}\xi^n-$$

$$\sum_{n=0}^{\infty}\sum_{k=0}^{q2}\alpha\eta^2\frac{(n-k+1)(n+1)}{(k+1)}B_k C_{n-k+1}\xi^n+\sum_{n=0}^{\infty}\sum_{k=0}^{q2}\left(\eta^2\frac{(n-k)(n+1)}{(k+2)}-\lambda\right)B_k C_{n-k}\xi^n$$

$$\ldots\ldots\ldots\ldots\ldots \quad (4.18)$$

Using equations (4.10) and (4.18) in the system equation eq(4.3). In its simplest form the system equation is reduced to :-

$$\sum_{n=0}^{\infty}\sum_{k=0}^{q1}(n-k+4)(n-k+3)(n+2)(n+1)A_k C_{n-k+4}\xi^n -$$
$$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{m}B_k\left(\frac{\alpha}{k+1}+\frac{1}{k+2}\right)+\mu(\alpha+1)\right)\eta^2(n+2)(n+1)C_{n+2}\xi^n +$$
$$\sum_{n=0}^{\infty}\sum_{k=0}^{q2}\alpha\eta^2\frac{(n-k+1)(n+1)}{(k+1)}B_k C_{n-k+1}\xi^n +$$
$$\sum_{n=0}^{\infty}\sum_{k=0}^{q2}\left(\eta^2\frac{(n-k)(n+1)}{(k+2)}-\lambda\right)B_k C_{n-k}\xi^n = 0$$

.... (4.19)

where q1 and q2 as defined previously.

Taking the summation with respect to n in eq(4.19) out :-

$$\sum_{n=0}^{\infty}\left(\sum_{k=1}^{q1}(n-k+4)(n-k+3)(n+2)(n+1)A_k C_{n-k+4} -\right.$$
$$\left(\sum_{k=0}^{m}B_k\left(\frac{\alpha}{k+1}+\frac{1}{k+2}\right)+\mu(\alpha+1)\right)\eta^2(n+2)(n+1)C_{n+2} +$$
$$\left.\sum_{k=0}^{q2}\alpha\eta^2\frac{(n-k+1)(n+1)}{(k+1)}B_k C_{n-k+1}+\sum_{k=0}^{q2}\left(\eta^2\frac{(n-k)(n+1)}{(k+2)}-\lambda\right)B_k C_{n-k}\right)\xi^n = 0$$

.... (4.20)

In order for this equation to be satisfied the terms inside the summation over n should be zero i.e.

$$\sum_{k=0}^{q1}(n-k+4)(n-k+3)(n+2)(n+1)A_k C_{n-k+4} -$$
$$\left(\sum_{k=0}^{m}B_k\left(\frac{\alpha}{k+1}+\frac{1}{k+2}\right)+\mu(\alpha+1)\right)\eta^2(n+2)(n+1)C_{n+2} +$$
$$\sum_{k=0}^{q2}\alpha\eta^2\frac{(n-k+1)(n+1)}{(k+1)}B_k C_{n-k+1}+\sum_{k=0}^{q2}\left(\eta^2\frac{(n-k)(n+1)}{(k+2)}-\lambda\right)B_k C_{n-k} = 0$$

n=0,1,2,3,... .... (4.21)

If solving for the largest coefficient C from the first term in the first series (k=0). The recurrence formula or equation becomes :-

$$C_{n+4} = \frac{-n!}{(n+4)!}\left(\sum_{k=1}^{q1} (n-k+4)(n-k+3)(n+2)(n+1)A_k C_{n-k+4} - \right.$$
$$\left(\sum_{k=0}^{m} B_k\left(\frac{\alpha}{k+1}+\frac{1}{k+2}\right)+\mu(\alpha+1)\right)\eta^2(n+2)(n+1)C_{n+2} +$$
$$\left.\sum_{k=0}^{q2}\alpha\eta^2\frac{(n-k+1)(n+1)}{(k+1)}B_k C_{n-k+1} + \sum_{k=0}^{q2}\left(\eta^2\frac{(n-k)(n+1)}{(k+2)}-\lambda\right)B_k C_{n-k}\right)$$

$n=0,1,2,3..;\infty$ .... (4.22)

From this equation, It is observed that the recurrence formula depends on the order of the series (m) which represents the cross-section properties variation. So It is very difficult to obtain explicite expressions for the coefficient (C_n) in terms of the arbitrary coefficients (C_0, C_1, C_2, and C_3) without knowing (m). Because of this, a solution for general cross-section properties variation can not be found analytically. Also if the cross-section properties variation series order (m) is knwon . The recurrence formula is nearly always very complex so that it can not be solved except for very simple cases like a constant cross-section variations (m=0) and the linearly tapered cross-section properties variation (m=1). Also noting that the problem under consideration is a vibration problem which includes one additional variable i.e. the non-dimensional natural frequency. The approximating function coefficients should be obtained in term of the natural frequency. This makes the problem very difficult to be solved analytically.

From this it is clear that computerization is the best way to solve this problem. The computer program shown in Appendix(B) solves this problem. The idea used in this program is to assume a value for the natural frequency. The recurrence formula is used to get the coefficients of the approximating function in terms of the arbitrary coefficients. The solution (W) and its first three derivatives are defined in terms of the arbitrary constants. By applying the boundary conditions a system of four homogeneous algebraic equations is obtained. This system can be written in matrix form as:-

$$[a_{ij}][C_i] = \{0\} \qquad \text{.............. (4.23)}$$

To have a solution for this system the determinate of the matrix [a] should be zero. if the determinate of the matrix [a] is zero the assumed value of the non-dimensional natural frequency is an actual non-dimensional natural frequency. If the determinant of the boundary conditions matrix is not zero the assumed non-dimensional natural frequency is not a real non-dimensional natural frequency and a new value for the non-dimensional natural frequency should be tried. In other words a search on the non-dimensional natural frequencies should be made.

After getting the natural frequencies . The boundary conditions are applied for the estimated non-dimensional natural frequency. and the boundary conditions matrix is found. By assuming one of the arbitrary coefficients to be one, the rest of the arbitrary coefficients are found in term of that assumed one. Using these coefficients the function W is written as a single function with one arbitrary variable.

In Appendix (B) a computer program which solves this problem is presented. Figure (B.1) shows a flow chart for this program, The program is explained and a sample run for this program is also shown.

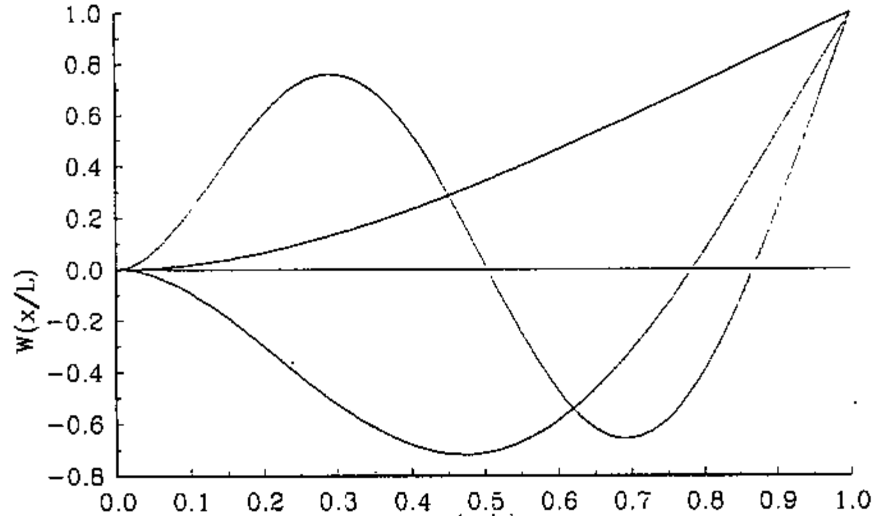
4.3 Numerical Study for the Results Found by the Series Method

As it was done in the solution by the finite element method a general study of the numerical results found by the computer program described in the previous section is first conducted. The two cases studied in reference [15]; namly the uniform cross-section and the linearly tapered cases will be used for comparison.

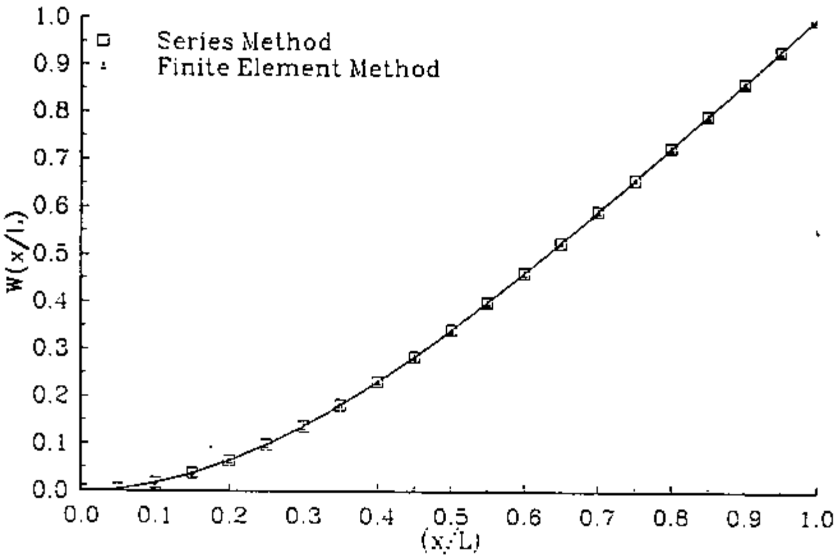
Table(4.1) shows the first four non-dimensional natural frequencies for the fixed-root, uniform cross-section beam with zero tip mass ratio, zero setting angle and zero root offset ratio using thirty five approximating function order with tolerance (EPS) equal to 1.0×10^{-6}. The results found in Ref[15] for the same case is also shown in this table.

Natural Frequ. No.	Exact Values From Ref[15]			Values From the Series Method Approximating Function Order equal 35		
Non-Dimensional Natural Frequency $(\omega\sqrt{(\rho A_0 L^4)/(EI_0)})$						
	$\eta=1$	$\eta=2$	$\eta=3$	$\eta=1$	$\eta=2$	$\eta=3$
1	3.6817	4.1373	4.7973	3.6187	4.1373	4.7973
2	22.1810	22.6149	23.3203	22.1810	22.6149	23.3203
3	61.8148	62.2732	62.9850	61.8148	62.2732	62.9850
4	121.051	121.497	122.236	121.051	121.497	122.236

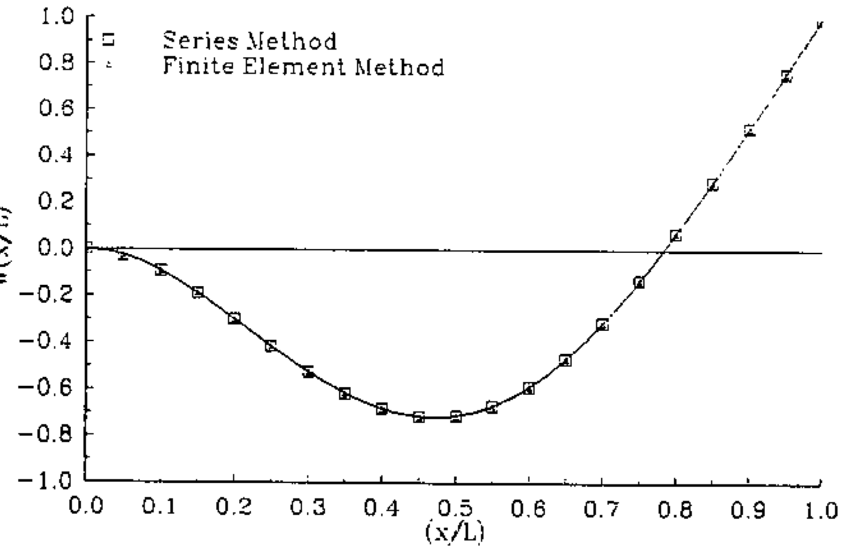
Table(4.1) Comparison of the series method results with the exact results, for uniform , fixed-root beam with $\mu=0$, $\theta=0$, $\alpha=0$, FORD=35, EPS= 1.0×10^{-6}



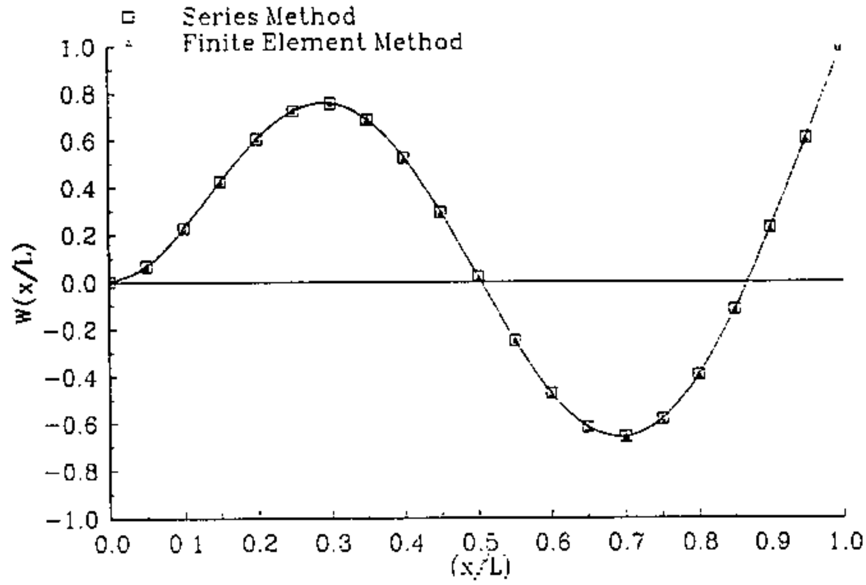
Fig(4.1) The first three mode shapes for uniform, fixed-root beam with $\eta=1.0$ $\mu=0$, $\theta=0$, $\alpha=0$, FORD=35, EPS= 1.0×10^{-6} using the series method



Fig(4.2) Comparison of the first mode shape found by using the series method with that found by the finite element method for fixed root uniform cross section beam with η=1.0 μ=0, θ=0, α=0



Fig(4.3) Comparison of the second mode shape found by using the series method with that found by the finite element method for fixed root uniform cross section beam with η=1.0 μ=0, θ=0, α=0



Fig(4.4) Comparison of the third mode shape found by using the series method with found by the finite element method for fixed root uniform cross section beam with η=1.0 μ=0, θ=0, α=0

From this table it is clear that the results approximated to four significant figures are the same as that found in Ref[15]. Fig(4.1) shows the first three mode shapes obtained for this case with non-dimensional rotational speed equal one. From this figure it is noted that the general shape of the first three mode shapes is in excellent agreement with that shown in Ref[15]. Regarding the mode shapes found by the series method and that found by using the finite element method. Figures (4.2), (4.3) and (4.4) show a comparison of the first three mode shapes obtained by using the series method with that obtained from the finite element method for the beam considered above with non-dimensional rotational speed equale one . It is interesting to note that for each figure the two curves can not be distinguished from each other.

The agreement between the finite element method and the series method results encourages the adoption of the finite element solution as being sufficiently accurate for the purpose of comparison .

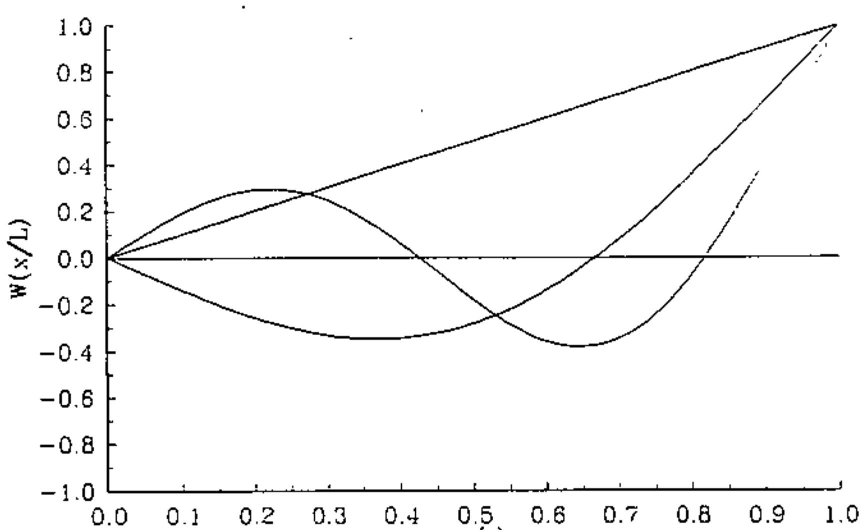
For the linearly tapered cross-section beam. The case chosen For comparison is the hinged root beam with zero tip mass ratio. zero setting angle and zero root offset ratio. A thirty five approximating function order and a tolerance (EPS) $=1.0\times10^{-6}$ were used .

Non-Dimensional Natural Frequency $(\omega\sqrt{(\rho A_0 L^4)/(EI_0)})$						
Natural Frequ. No.	Exact Values From Ref[15]			Values From the Series Method Approximating Function Order equal 35		
	$\eta=1$	$\eta=2$	$\eta=3$	$\eta=1$	$\eta=2$	$\eta=3$
1	1.0000	2.0000	3.0000	1.0000	2.0000	3.0000
2	16.8711	17.2794	17.9388	16.8711	17.2794	17.9388
3	48.5872	48.9395	49.5210	48.5872	48.9395	49.5210
4	97.2823	97.6172	98.1727	97.2823	97.6172	98.1727

Table(4.2) Comparison of the series method result with the exact results, for tapered , hinged-root beam with with $\eta=1.0$ $\mu=0$, $\theta=0$, $\alpha=0$, FORD=35, EPS= 1.0×10^{-6} using the series method , for which :

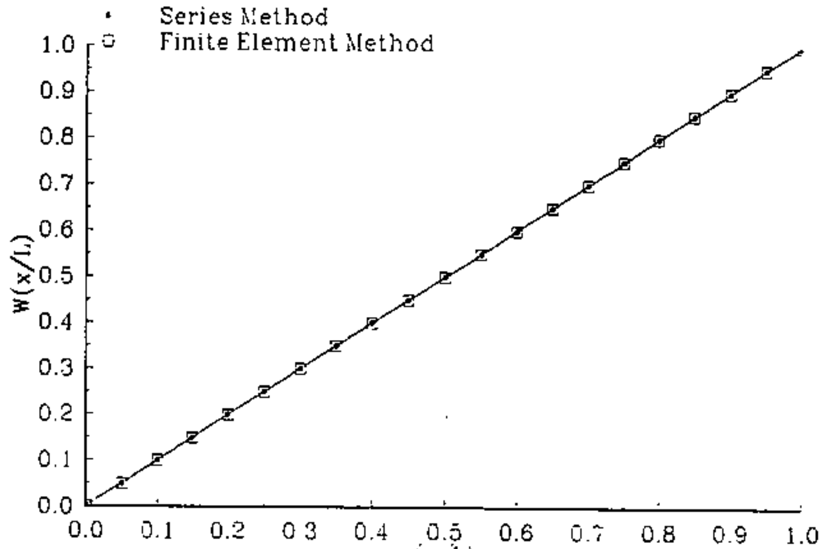
$$f(\xi)=1-0.95\xi$$
$$g(\xi)=1-0.80\xi$$

For this case the results are shown in table(4.2). From this table it is noted that for four significant figures the results are the same as those obtained from Ref[15]. Fig(4.5) shows the first three mode shapes for the case under consideration with non-dimensional rotational speed equal two.



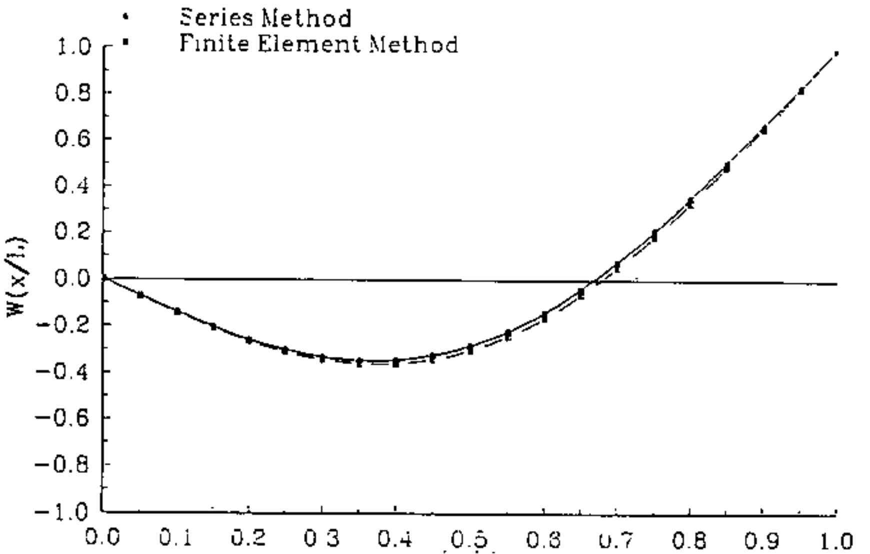
Fig(4.5) The first three mode shapes for tapered, hinged-root beam with $\eta=2.0$, $\mu=0$, $\theta=0$, $\alpha=0$, FORD=35, EPS= 1.0×10^{-6} using the series method for which:-

$$f(\xi)=1-0.95\xi$$
$$g(\xi)=1-0.80\xi$$



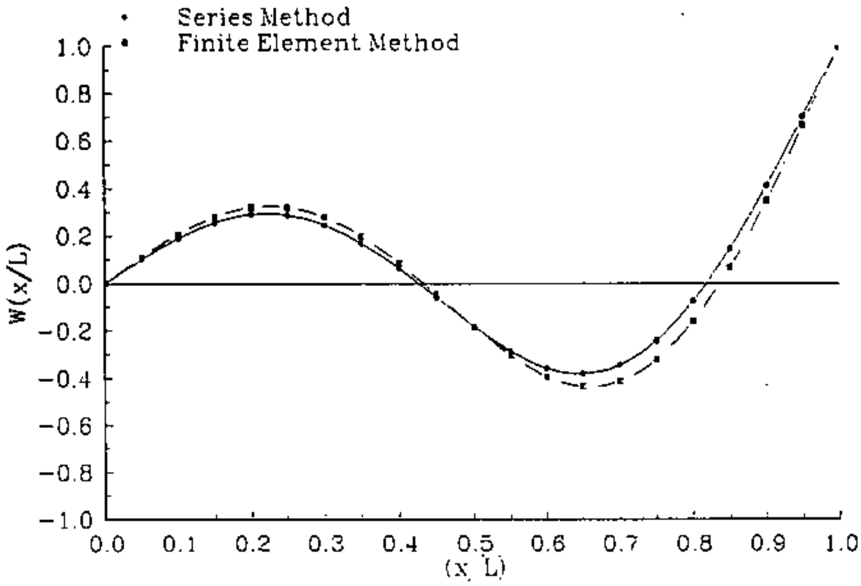
Fig(4.6) Comparison of the first mode shape found by using the series method with that found by the finite element method for tapered, root hinged cross section beam with η=2.0 μ=0, θ=0, α=0 for which

$$f(\xi) = 1 - 0.95\xi$$
$$g(\xi) = 1 - 0.80\xi$$



Fig(4.7) Comparison of the second mode shape found by using the series method with that found by the finite element method for hinged root tapered cross section beam with η=2.0 μ=0, θ=0, α=0 for which

$$f(\xi) = 1 - 0.95\xi$$
$$g(\xi) = 1 - 0.80\xi$$



Fig(4.8) Comparison of the third mode shape found by using the series method with that found by the finite element method for hinged root tapered cross section beam with $\eta=2.0$ $\mu=0$, $\theta=0$, $\alpha=0$ for which

$$f(\xi) = 1 - 0.95\xi$$
$$g(\xi) = 1 - 0.80\xi$$

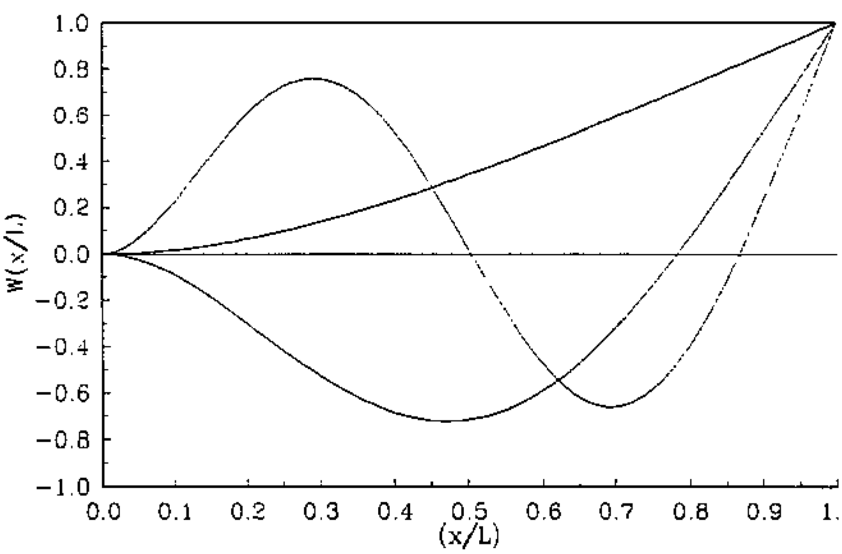
For the beam under consideration Figures (4.6), (4.7) and (4.8) show a comparison of the first three mode shapes obtained from the series method with that obtained by using the finite element method. From these figures it is clear that the error in the mode shape depends on the number of the mode shape. If the number of the mode shape higher, larger error is observed. Also from the above two cases it is noted that increasing the cross-section properties variation series order will increase the error in the estimated natural frequencies and the mode shapes..

The series method can also be used for non-rotating beams by just setting the non-dimensional rotational speed (η) to zero. For example the non-dimensional natural frequencies of a uniform cross-section non-rotating beam with zero tip mass, zero setting angle, and zero root offset ratio are shown in table(4.3). The exact values found in Ref[15] are also shown. For four significant figures, the results are the same and have no error.

Non-Dimensional Natural Frequency $(\omega\sqrt{(\rho A_0 L^4)/(EI_0)})$		
No	Series Method Using Approximating Function Order equal Thirty five	Exact Values
1	3.5160	3.5160
2	22.0345	22.0345
3	61.6972	61.6972
4	120.902	120.902

Table(4.3) The first five non-dimensional natural frequencies for uniform non-rotating , fixed-root beam with η=2.0 μ=0, θ=0, α=0, FORD=35, EPS= 1.0×10^{-6}

Fig(4.9) shows the first three mode shapes for the non-rotating fixed root uniform cross section beam considered above. When the mode shapes are compared with those found using the finite element method, each two corresponding mode shape curves didn't show any difference as those for the uniform, fixed root beams shown above. So there is no need to show them.



Fig(4.9) The first three mode shapes for unifier, fixed-root beam with η=2.0 μ=0, θ=0, α=0, FORD=35, EPS= 1.0×10^{-6} using the series method

The running time of the program depends on the approximating function order (FORD), the tolerance (EPS), the cross-section properties variation series order (SORD) and the number of natural frequencies required (NEV) in addition of its being machine dependent i.e. It also depend on the speed of the computer used. Most of the time is consumed in the search for the root intervals during the execution of the function search( ). The running time varies from less than one minute to about half an hour depending on the case under consideration.

The error in the estimated natural frequencies and mode shapes using the series method depend on two factors. These are :-

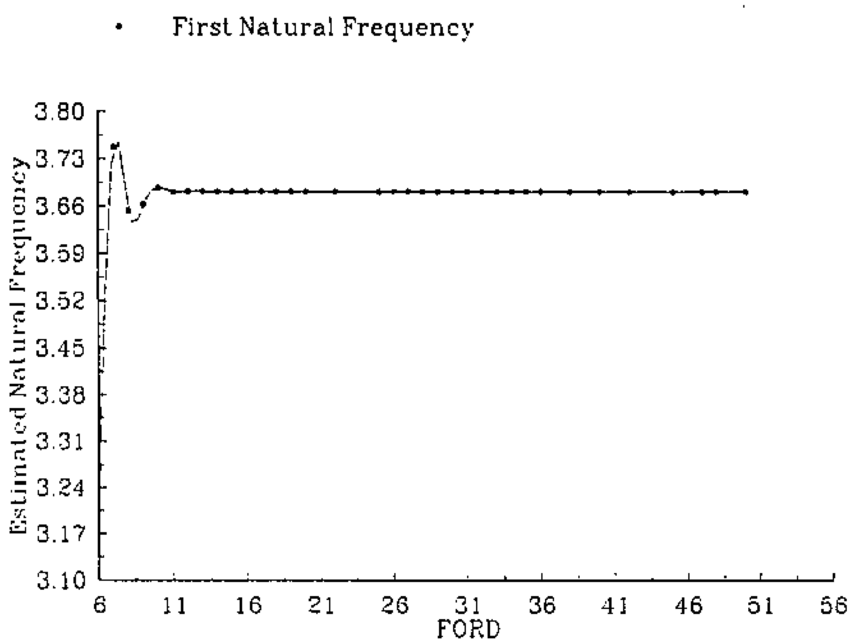
1- The approximating function order (FORD).

2- The tolerance (EPS).

In the following two section a study of the effect of these factors will be presented.

4.3.1 The Effect of the Approximating Function Order (FORD)

The approximating function order (FORD) shows a behavior which is difficult to be explained, first of all it is always very difficult to say anything about the number of non-dimensional natural frequencies that can be found for a certain order of the approximating function. Because of that for the approximating function order under consideration the program is to be run for a certain number of non-dimensional natural frequencies. If overflow happened then the number of non-dimensional natural frequencies required should be reduced until results are obtained. On the other hand if results are obtained, the number of non-dimensional natural frequencies required should be increased. The maximum number of non-dimensional natural frequencies found before overflow is the maximum number of non-dimensional natural frequencies that can be found for the approximating function order under consideration.



Fig(4.10) The effect of the approximating function order on the estimated first non-dimensional natural frequency for fixed root uniform cross section beam with η=1.0 μ=0, θ=0, α=0, EPS= 1.0×10^{-6}

In some cases increasing the approximating function order by one causes the program to reach overflow before getting the highest natural frequency that the program can estimate for the previously used approximating function order, and not only this but also the error in the estimated natural frequencies increases. It was noted that by increasing the approximating function order by at most three an approximation for a new natural frequency can be found.

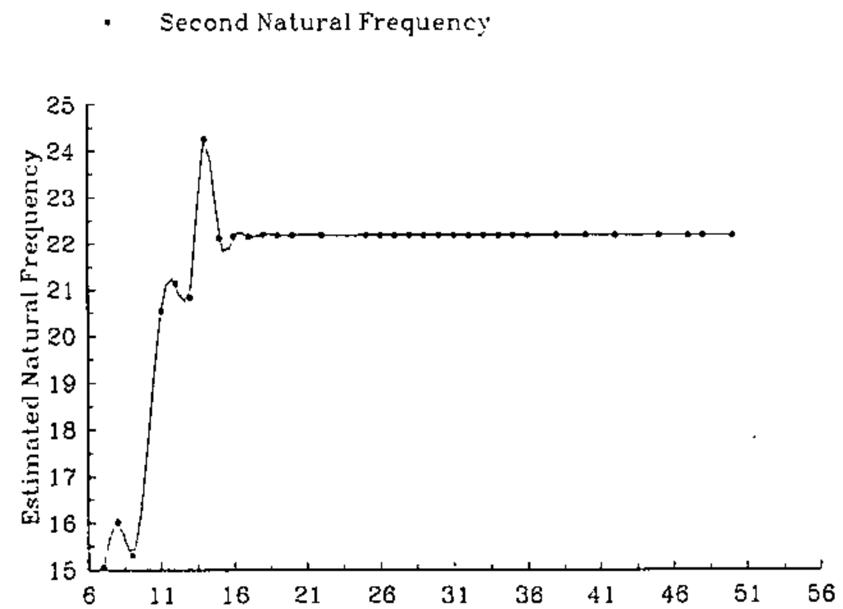
Other cases increasing the order of the approximating function doesn't allow finding a new natural frequency. At the same time it improves the estimated non-dimensional natural frequencies previously estimated. In some cases the increase in the approximating function order doesn't do any thing .

This behavior is unpredictable. But regarding the added coefficients it may expected that :-

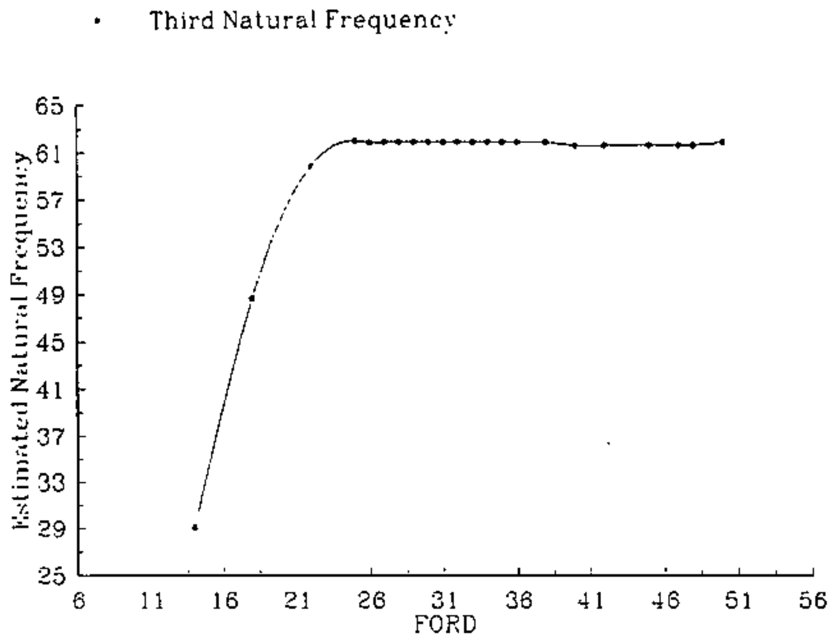
1- The added coefficient is zero or very small so that it will not affect the roots of the polynomial.

2-The added coefficient has a value which shifts one of the root of the approximating polynomial to a negative value. So it can not be estimated because the search is done in the positive region, this is concluded because in the cases a root is missed. It is noted that anew non-dimensional natural frequency can always be found after increasing the approximating function order by at most three.

3-In the third case it is expected that the added term to the polynomial has a value which tends to improve the estimated non-dimensional natural frequencies but not as important as those which causes one of the estimated natural frequency to disappear.

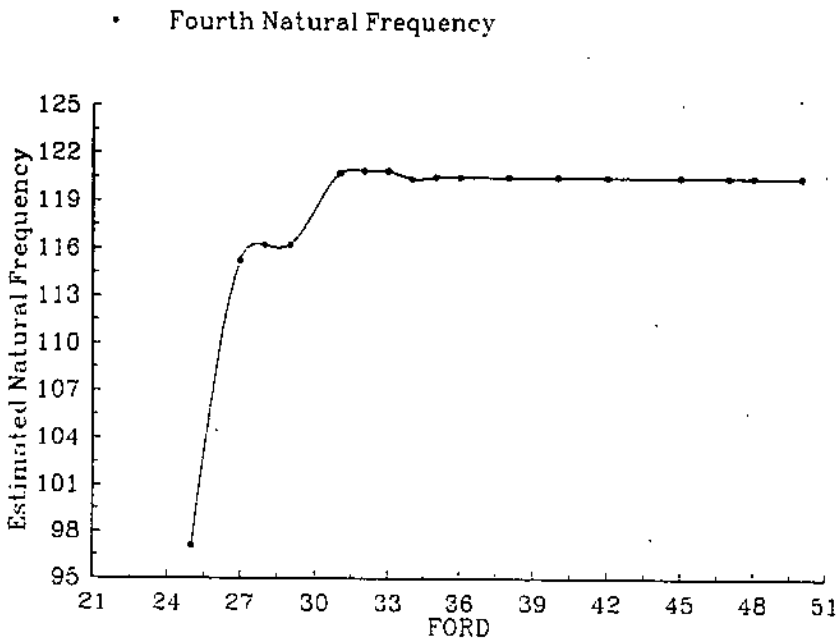


Fig(4.11) The effect of the approximating function order on the estimated second non-dimensional natural frequency for fixed root uniform cross section beam with η=1.0 μ=0, θ=0, α=0, EPS= 1.0×10^{-6}



Fig(4.12) The effect of the approximating function order on the estimated third non-dimensional natural frequency for fixed root uniform cross section beam with η=1.0 μ=0, θ=0, α=0, EPS= 1.0×10^{-6}

Figures (4.10), (4.11), (4.12) and (4.13) show the effect of the approximating function order (FORD) on the estimated non-dimensional natural frequencies for a fixed root uniform cross-section beam with zero tip mass ratio, zero setting angle and zero root offset ratio with non-dimensional rotational speed (η) equal one and tolerance (EPS) 1×10^{-6}. From these figures it is noted that the minimum approximating function order for which a non-dimensional natural frequency can be found is six, less than this order no non-dimensional natural frequency can be found. Also for each non-dimensional natural frequency there is a minimum value of the approximating function order below which no approximation for it can be found. For this approximating function order the non-dimensional natural frequency obtained is just an approximation and it always has large error. To improve this obtained approximation of the non-dimensional natural frequency the approximating function order should be increased.

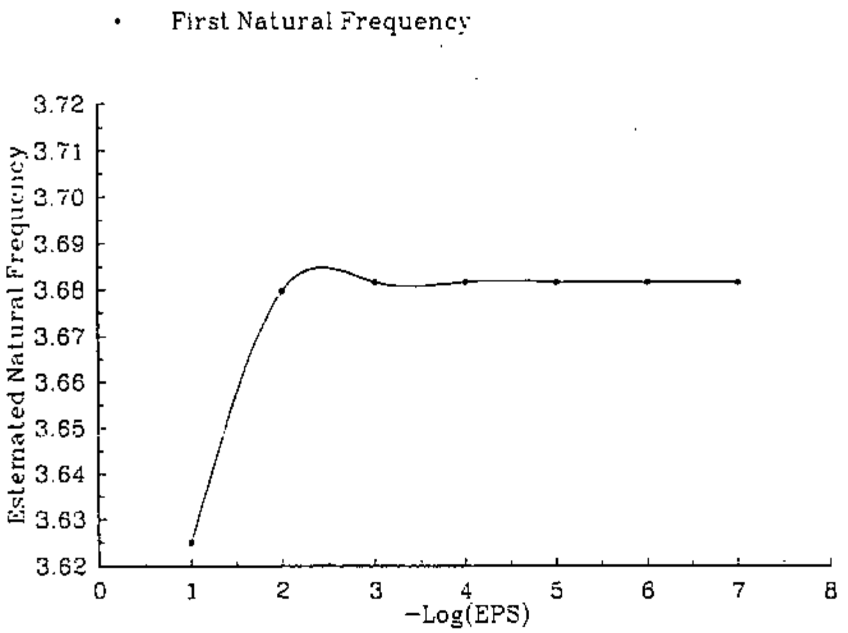


Fig(4.13) The effect of the approximating function order on the estimated fourth non-dimensional natural frequency for Fixed root uniform cross section beam with η=1.0 μ=0, θ=0, α=0, EPS= 1.0×10^{-6}

Also from this study. It is noted that for each non-dimensional natural frequency estimated, at first it will encounter some error. By increasing the approximating function order the error in the estimated non-dimensional natural frequency decrease. Also it was noted that by increasing the function order by not more than five an exact value of the non-dimensional natural frequency is obtained. Stile for some function order the non-dimensional natural frequency disappears once. After that the non-dimensional natural frequency reaches a steady value That value is the exact value of the non-dimensional natural frequency.

4.4.2 The Effect of the Tolerance (EPS)

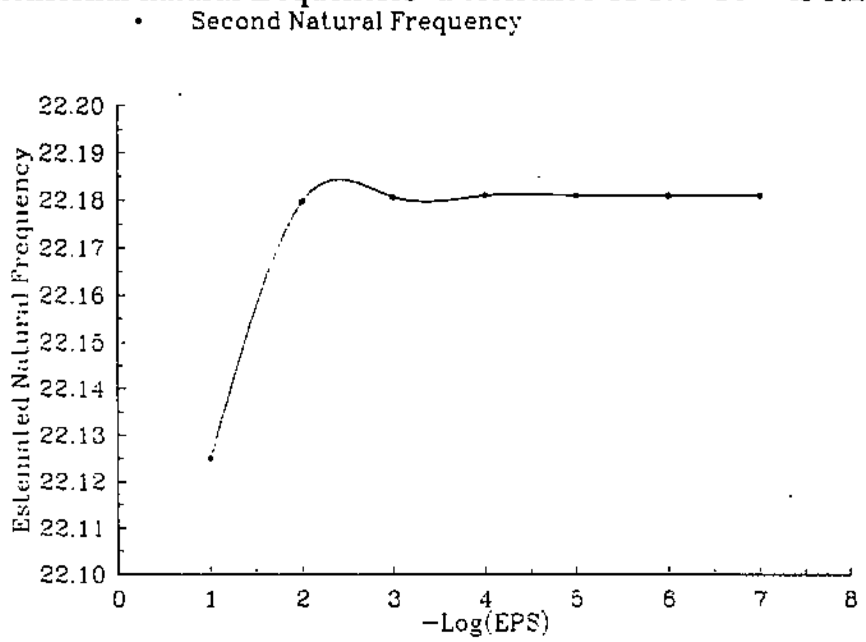
The effect of the tolerance (EPS) is much more clear than the effect of the approximating function order. This is because in programming the bisection method was used to find the non-dimensional natural frequency within an intervals whic was previously specified using the function search( ). Because of this in each



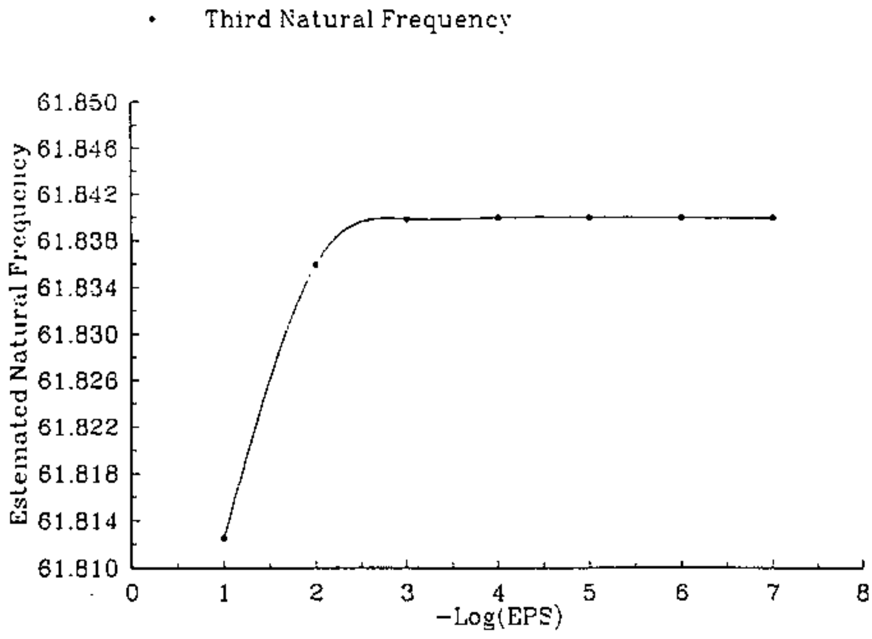
Fig(4.14) The effect of the tolerance(EPS) on the estimated first non-dimensional natural frequency for a fixed root uniform beam with $\eta=1.0$ $\mu=0$, $\theta=0$, $\alpha=0$, FORD=35

case considered the non-dimensional natural frequency is always specified within an interval. The width of this interval is equal the tolerance(EPS) and the maximum error expected is equal to half the tolerance(EPS).

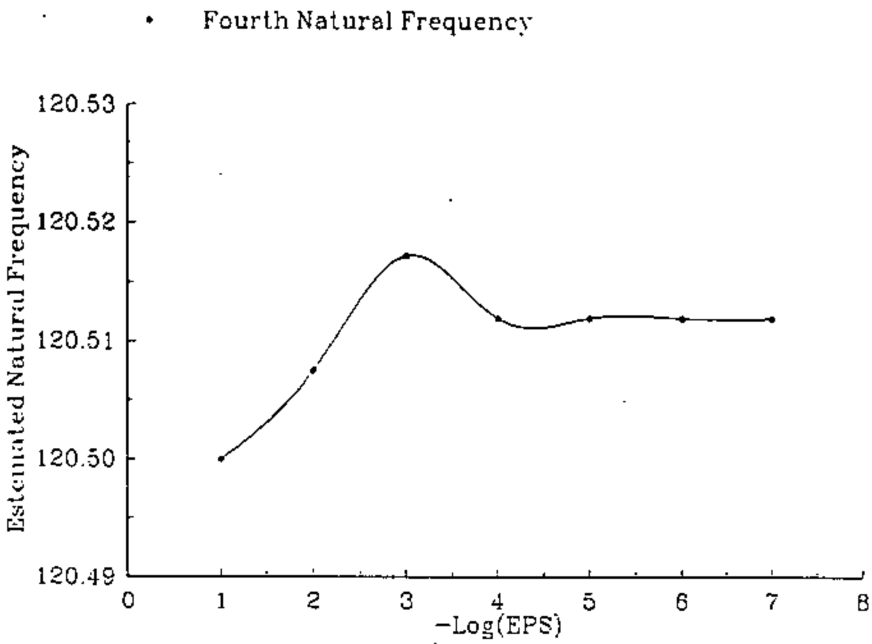
Figures (4.14), (4.15), (4.16), and (4.17) show the effect of the tolerance on the first four non-dimensional natural frequencies. The case considered is a fixed root uniform cross-section beam with zero tip mass ratio, zero setting angle and zero root offset ratio with non-dimensional rotational speed(η) equal one. In the solution an approximated function order (FORD) of thirty five was used. It is clear from these figures that since a four significant figures approximation is required in approximating the non-dimensional natural frequencies a tolerance of 1.0×10^{-5} is sufficient.



Fig(4.15) The effect of the tolerance (EPS) on the estimated second non-dimensional natural frequency for a fixed root uniform beam with η=1.0 μ=0, θ=0, α=0, FORD=35



Fig(4.16) The effect of the tolerance (EPS) on the estimated third non-dimensional natural frequency for a fixed root uniform beam with η=1.0 μ=0, θ=0, α=0, FORD=35



Fig(4.17) The effect of the tolerance (EPS) on the estimated fourth non-dimensional natural frequency for a fixed root uniform beam with $\eta=1$, $\mu=0$, $\theta=0$, $\alpha=0$, FORD=35

From this study it was found that for any considered case, if four natural frequencies are required with four significant figures approximation. Using approximating function order of thirty five and a tolerance (EPS) of 1.0×10^{-6} are sufficient to get an excellent solution.

Chapter Five

Parameteric Study for the Problem of Free Vibration of Rotating Beam with Non-Uniform Cross-Section

5.1 Introduction

In this chapter a parameteric study for all parameters that affect the problem of free vibration of rotating beams with non-uniform cross-section properties will be conducted using the developed computer programs shown in appendices (A) and (B). The study will be conducted using the series method program. The finite element method program will be used for comparing some cases to insure that the series method program works well.

The parameters which will be studied are thoes expected to affect the non-dimensional natural frequencies and mode shapes. Thoes parameters are :-

1- The non-dimensional rotational speed (η)

2- The tip mass ratio (μ)

3- The root offset ratio (α)

4- The setting angle (θ)

5- The translation stiffness ratio (β)

6- The rotational stiffness ratio (γ)

The cross-section properties variation series can be of any order. No cross-section properties variation has any advantage on other cases in the analysis. Also the flexural rigidity series and mass per unit length series are not necessary to be of the same order. If it happens that the flexural rigidity series and mass per unit length series are of different orders, then the higher order of the two series should be used and the new high coefficients of the lower order series should be set to zero.

The uniform cross-section and the linearly tapered cross-section cases have been studied in Ref[15]. They are also used in chapters three and four for the study of the numerical results obtained from the computer programs developed. The cross-section properties variation case which will be used in the parametric study is expected to be a new case which has not been studied befor. At the same time this case was decided to be of practical value. So the exponentially decaying function is chosen for this study which is given by:-

$$f(\xi) = g(\xi) = e^{-\xi} \quad \text{..............} \quad (5.1)$$

This case will illustrate the way to deal with cross-section variations given in the form of analytical functions.

For any cross-section properties variations given in analytical form the Taylor series expansion can be employed to get a power series approximation. The Taylor series approximation is given by :-

$$F(x) = \sum_{n=0}^{\infty} \frac{F^{(n)}(0)}{n!} x^n \quad \text{..............} \quad (5.2)$$

where $F^{(n)}(0)$ is the n'th derivetive of the function F(x) with respect to x at x=0.

For the case considered, the Taylor series approximation is:-

$$f(\xi) = g(\xi) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \xi^n \quad \text{..............} \quad (5.3)$$

Next, it is required to find the order of truncated Taylor series which will give acceptable representation for this case. This order is the one for which increasing the order of Taylor series approximating more will not affect the estimated non-dimensional natural frequencies. Table(5.1) shows the first four non-dimensional natural frequencies for fixed root, exponentially decaying cross-section properties variation beam with zero tip mass ratio, zero setting angle, zero root offset ratio and non-dimensional rotational speed equal one using the series method with thirty five approximating function order and tolerance (EPS) equal 1.0×10^{-6}.

Non-Dimensional Natural Frequencies Using The Series Method				
SORD	w1	w2	w3	w4
3	4.95771	24.54086	64.29287	125.73225
4	4.83331	24.23000	63.81591	122.12307
5	4.86580	24.23143	63.98470	122.76270
6	4.86285	24.31305	63.97055	122.73309
7	4.86323	24.31413	63.96873	122.69147
8	4.86372	24.31583	63.97990	122.68713
9	4.86319	24.31400	63.97595	122.77471
10	4.86319	24.31402	63.97269	122.79790
11	4.86319	24.31402	63.96942	122.75319
12	4.86319	24.31393	63.96923	122.73434
13	4.86319	24.31400	63.96942	122.7029
14	4.86319	24.31404	63.96897	122.72655
15	4.86319	24.31400	63.96942	122.70290
16	4.83619	24.31400	63.96942	122.70290
17	4.86319	24.31400	63.96942	122.70290
18	4.86319	24.31400	63.96942	122.70290

Table(5.1) The First four non-dimensional natural frequencies for a fixed root with exponentially varying cross-section beam with $\eta=1$, $\mu=0$, $\theta=0$, $\alpha=0$, FORD=35, EPS= 1.0×10^{-6} for different values of the SORD using the series method

from table(5.1) it is noted that for a five digit accuracy at least sixteen terms of the Taylor series approximation are needed to get results which are independant of the order of cross-section properties variation series for the considered case, also it is noted that the error in the approximation depends on the number of the highest non-dimensional natural frequency required. Based on the results shown in table(5.1) it is obsearved that using cross-section properties variation series order of eighteen gives an acceptable approximate for the cross-section properties variation considered.

In the parameteric study, the parameters to be studied will be varied over ranges which are expected to be physically valid. The non-dimensional speed will be varied over the range from one to ten. The tip mass ratio will be studied over the rang from zero to one half. The setting angle will be varied from zero degree to ninety degrees. the root offset ratio will be studied over the range zero to five. In studying the effect of the translation stiffness ratio a very high rotational stiffness ratio will be used so that it may be considered very stiff in rotation and the translation stiffness ratio will be varied from zero to the limiting case in which the root may be considered fixed with translation stiffness ratio is infinite. For the last parameter the rotational stiffness ratio, the translation stiffness ratio will be taken very high to insure that the beam under consideration can be considered very stiff in translation, and the rotational stiffness ratio will be varied from the limiting case with zero rotational stiffness i.e. hinged root case to the limiting case in which the rotational stiffness is infinite i.e. the fixed root case.

In the parameteric study some of the results will be presented in this chapter in a graphical form and all the results are presented in tabulated form in Appendix C.

5.2 The Effect of the Non-Dimensional Rotational Speed (η)

The non-dimensional rotational speed as defined in chapter two, incorporate factors that define the geometry of the beam and the properties of the beam material in addition to the motion itself.

The geometry of the beam appears through the ratio of the cross-section area to the second moment of area of the cross section at the root and the beam length. The material properties of the beam appears through the ratio of the density to the modlus of elasticity of the beam material. The motion effect appears through the rotational speed itself. Changing any of these parameters will change the non-dimensional rotational speed, so the study of the non-dimensional rotational speed will represents a study of all of these parametiers simultaneously.

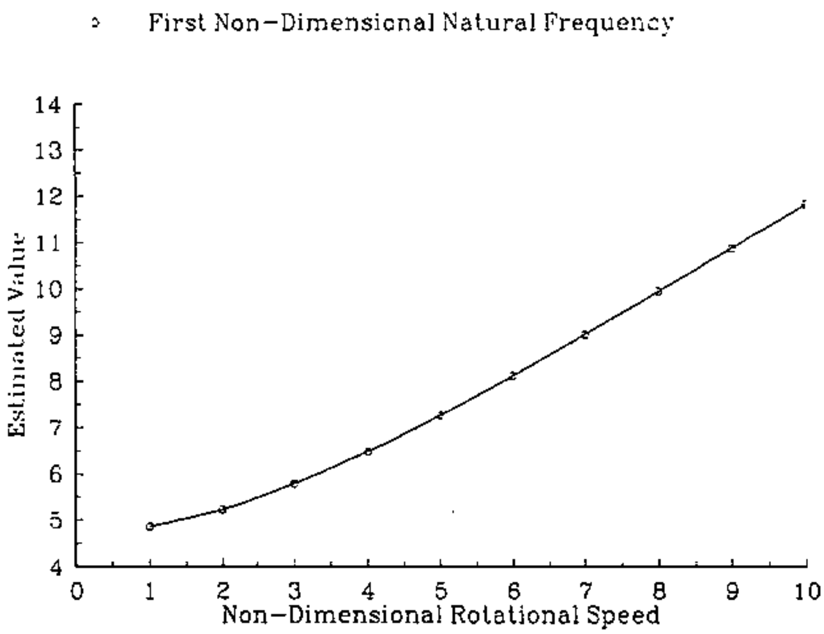
In this study the case of non-uniform cross-section properties which its cross-section properties are vary exponentially is used as stated in

Non-Dimensional Natural Frequency ($\omega\sqrt{(\rho A_0 L^4)/(EI_0)}$)				
Series Method with FORD=35 and EPS=1.0×10^{-6}			Finite Elemnt Method With NFE=15 and EPS=1.0×10^{-6}	
No.	Fixed Root Case	Hinged Root Case	Fixed Root Case	Hinged Root Case
1	4.6832	1.0000	4.8626	1.0003
2	24.3140	16.3661	24.3136	16.2712
3	63.9694	51.2332	63.9766	51.2407
4	122.7029	105.4930	123.2336	105.5317

Table(5.2) Comparison of the results found by the series method with the finite element method results for an exponintially varying cross-section beam with $\eta=1$, $\mu=0$, $\theta=0$, $\alpha=0$.

section (5.1), using approximating function order of thirty five and tolerance of 1.0×10^{-6} .The non-dimensional rotational speed varied over the range zero to ten as stated in section (5.1), using approximating function order of thirty five and tolerance of 1.0×10^{-6} . The non-dimensional rotational speed was varied over the range zero to ten, this range is expected to be physically accepted. Most of the papers worked in this field used this range. This study is conducted for both fixed root and hinged root cases. Some of the results obtained are shown graphically in figures (5.1), (5.2), (5.3) and (5.4) and all the results obtained are shown in appendix C in tables (C..1) and (C.2).

The results found from the finite element method show excellent agreement with those found from the series method. Table(5.2) shows a comparison of the first four non-dimensional natural frequencies found by the finite element method with those found by the series method. These results



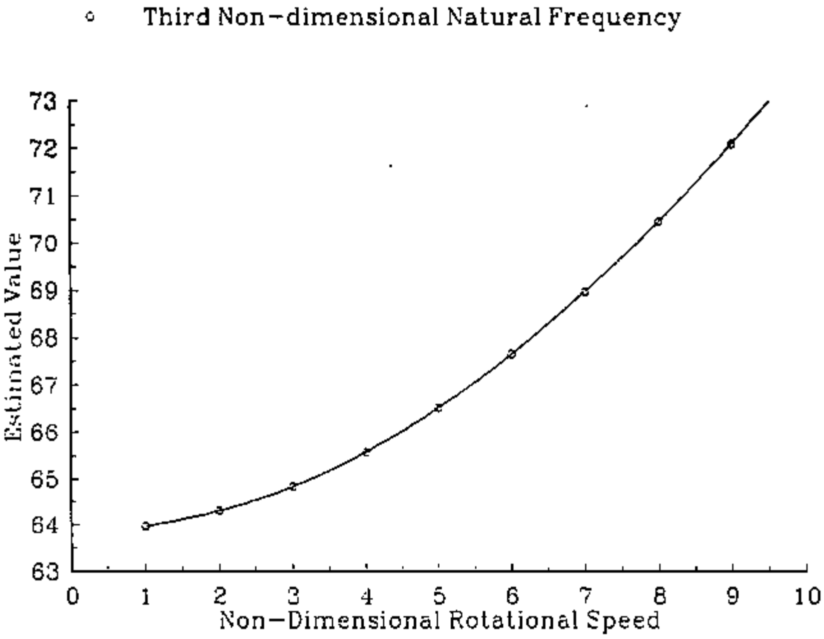
Fig(5.1) The effect of the non-dimensional rotational speed on the first non-dimensional natural frequency for fixed root exponentially varying cross-section beam with μ=0, θ=0, α=0, FORD=35, EPS= 1.0×10^{-6}

are obtained for the case previously described with both fixed root and hinged root conditions with zero tip mass ratio, zero setting angle and zero root offset ratio with non-dimensional rotational speed (η) equal one.

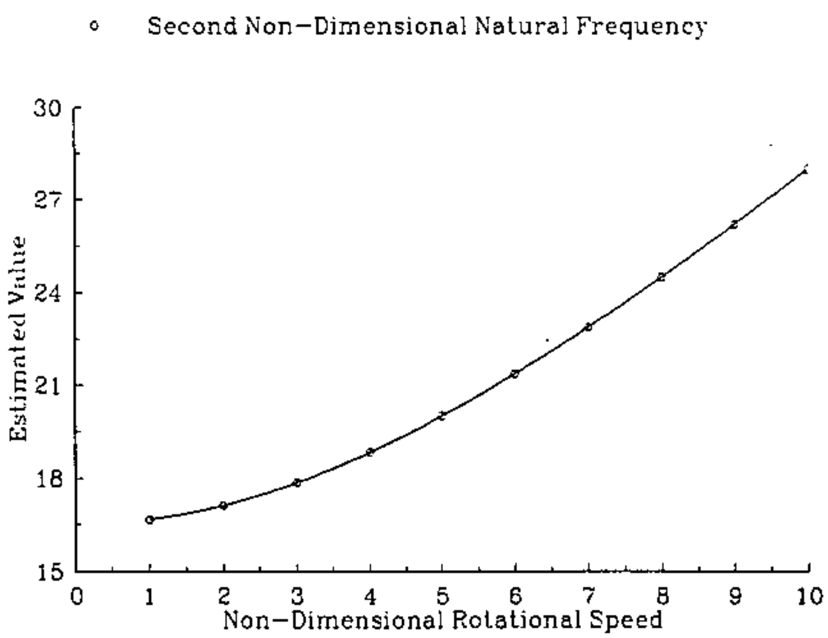
From these results, it is noted that the results found from the finite element method is higher in value than that found from the series method, and this is expected because the finite element method results are approximate results and the exact solution is the one which minimize the system energy. so the power series method results are expected to be less because it is the exact solution.

Figures (5.1) and (5.2) show the effect of the non-dimensional rotational speed on the first and third non-dimensional natural frequencies for fixed root beam, also Figures (5.3) and (5.4) show the effect of the non-dimensional rotational speed on the second and fourth non-dimensional natural frequencies for hinged root beam. In both cases the tip mass ratio, the setting angle and the root offset ratio are taken to be zero, from these figures it is noted that the increase in the non-dimensional rotational speed will increase the non-dimensional natural frequency for both cases, the effect of the non-dimensional rotational speed is more significant for small non-dimensional natural frequencies than the high ones. This effect is explained by the fact that the centrifugal force produced by the rotational increases the beam stiffness, and this causes the non-dimensional natural frequencies to increase.

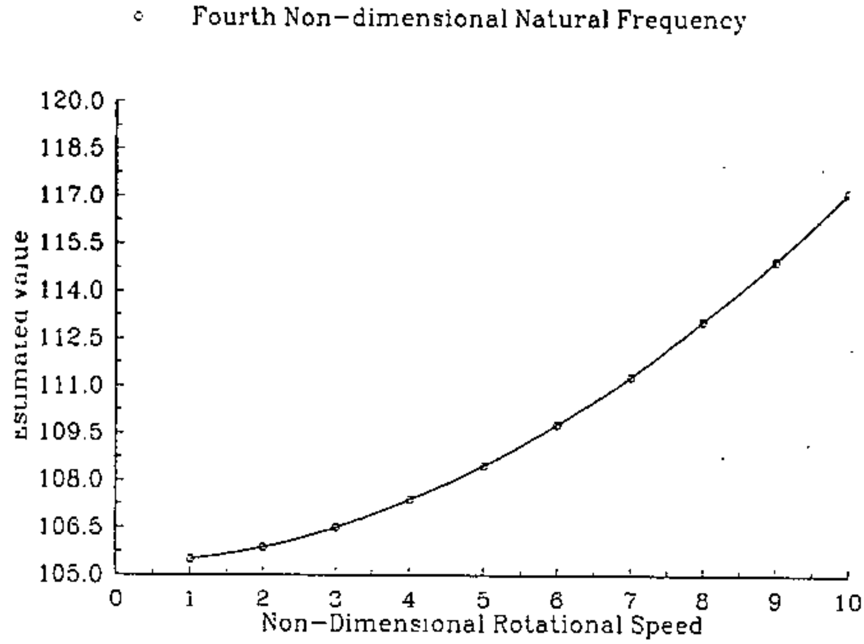
It is expected that the mode shapes will be affected by changing the non-dimensional rotational speed. Physically since increasing the non-dimensional rotational speed will increase the inertia force which is in the axial direction. this will force the curvature of the mode shape to decrease, however the change is small. This is also clear from Figure (5.5) which shows the first mode shape for



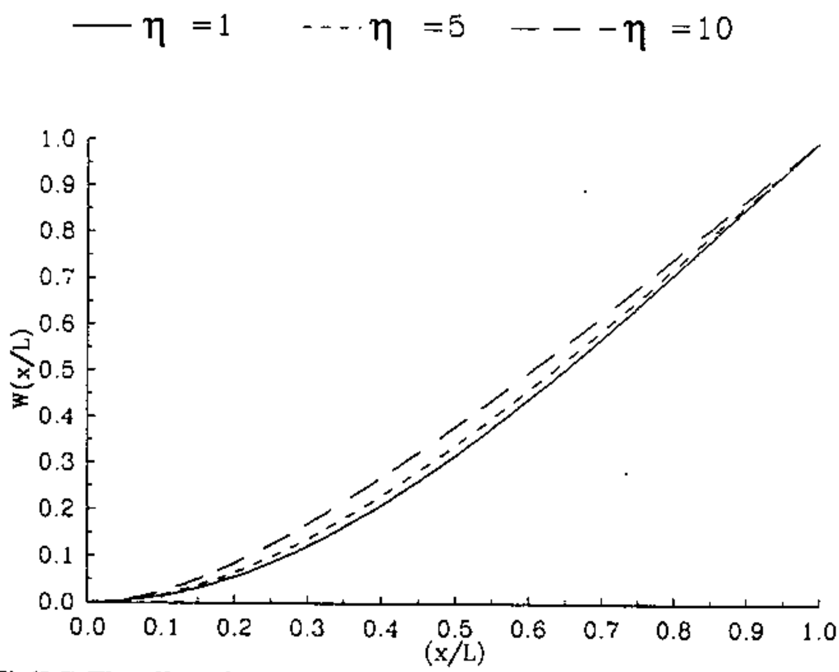
Fig(5.2) The effect of the non-dimensional rotational speed on the third non-dimensional natural frequency for fixed root exponentially varying cross-section beam with $\mu=0$, $\theta=0$, $\alpha=0$, FORD=35, EPS= 1.0×10^{-6}



Fig(5.3) The effect of the non-dimensional rotational speed on the second non-dimensional natural frequency for hinged root exponentially varying cross-section beam with $\mu=0$, $\theta=0$, $\alpha=0.$, FORD=35, EPS= 1.0×10^{-6}



Fig(5.4) The effect of the non-dimensional rotational speed on the fourth non-dimensional natural frequency for hinged root exponential cross-section beam with μ=0, θ=0, α=0. , FORD=35, EPS= 1.0×10^{-6}



Fig(5.5) The effect of the non-dimensional rotational speed on the first mode shape for fixed root exponentially varying cross-section beam with η=1, μ=0, θ=0, α=0, FORD=35, EPS= 1.0×10^{-6}

the non-dimensional rotational speed equal one, five and ten. These results are obtained for the fixed root case with zero tip mass ratio, zero setting angle and zero root offset ratio. This is true for the higher mode shapes with both fixed root and hinged root cases.

5.3 The Effect of the Tip Mass Ratio (μ)

The tip mass ratio represents the ratio of the tip mass to the mass of a uniform beam having the same length of the beam under consideration and a cross section of the considered beam at the root.

Non-Dimensional Natural Frequency $(\omega\sqrt{(\rho A_0 L^4)/(EI_0)})$				
Series Method with FORD=35 and EPS=1.0×10^{-6}			Finite Elemnt Method With NFE=15 and EPS=1.0×10^{-6}	
No.	Fixed Root Case	Hinged Root Case	Fixed Root Case	Hinged Root Case
1	2.25305	1.0000	2.25344	0.9998
2	17.7309	11.2391	17.7311	11.2409
3	52.3798	40.9271	52.4036	40.9318
4	106.7131	90.3086	106.9361	90.2809

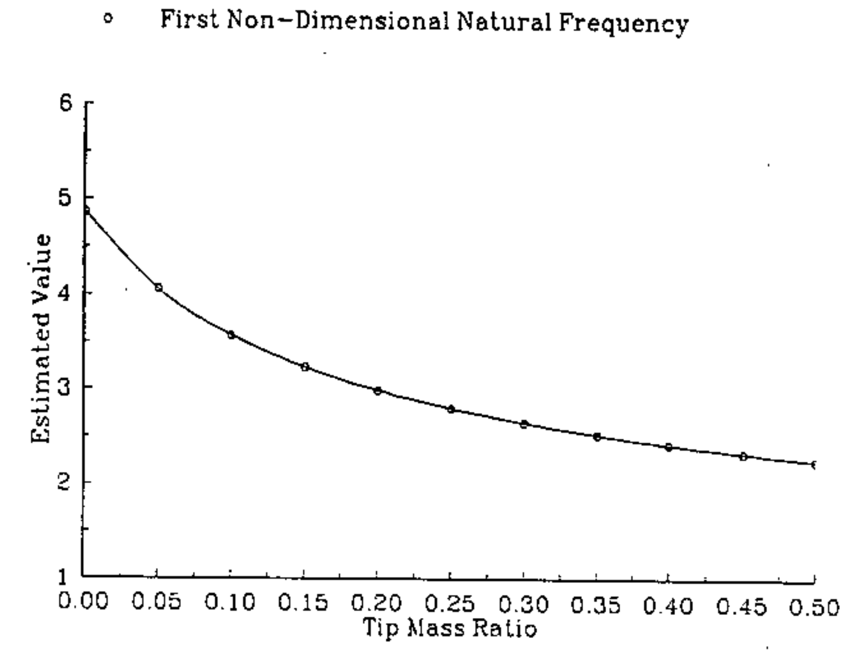
Table(5.3) Comparison of the results found by the series method with the finite element method results for an exponintially varying cross-section beam with η=1, μ=0.5, θ=0, α=0

Table(5.3) shows a comparison of the results obtained by the series method with those found from the finite element method for exponentially varying cross-section beam with zero setting angle, zero root offset ratio and non-dimensional rotational speed equal one, for both fixed root and hinged root cases with tip mass ratio equal 0.5. It is clear that the results in the table show an excellent agreement.

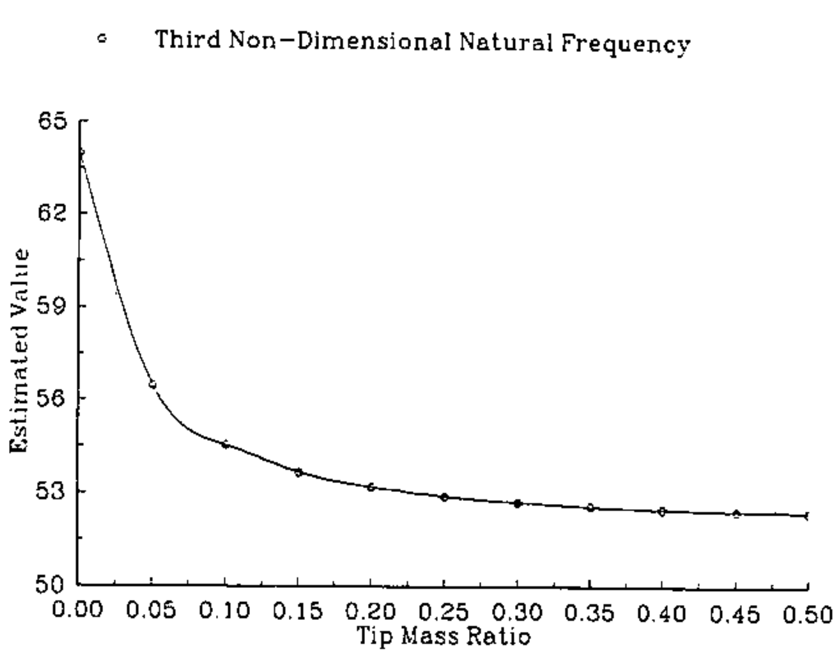
In the system equation the tip mass ratio appears in the inertia term and in the boundary conditions at the tip. So it is expected that the tip mass ratio will have more significant effect on the non-dimensional natural frequencies and mode shapes.

Figures (5.6) and (5.7) show the effect of the tip mass ratio on the first and third non-dimensional natural frequencies a fixed root exponentially varying cross section properties beam with zero setting angle, zero root offset ratio and non-dimensional rotational speed equal one using thirty five approximating function order and tolerance 1.0×10^{-6}, also figures (5.8) and (5.9) show the effect of the tip mass ratio on the second and fourth non-dimensional natural frequencies for the same case with hinged root.

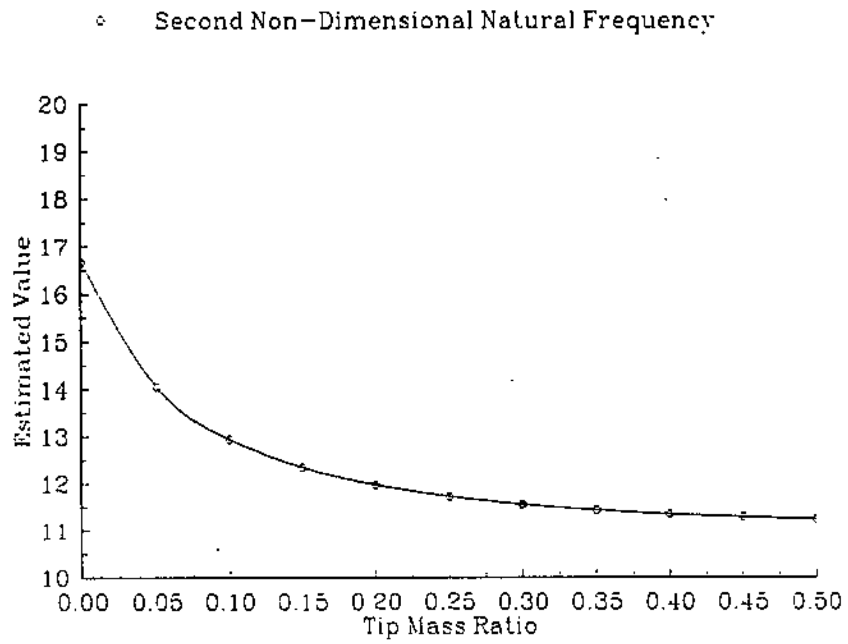
From these figures it is noted that the increase in the tip mass ratio tends to reduce the estimated non-dimensional natural frequency. For small tip mass ratios increasing the tip mass ratio leads to higher reduction in the estimated non-dimensional natural frequency than that for high tip mass ratio values, at the same time it is clear that for high non-dimensional natural frequencies the tip mass ratio has smaller effect than that for the low



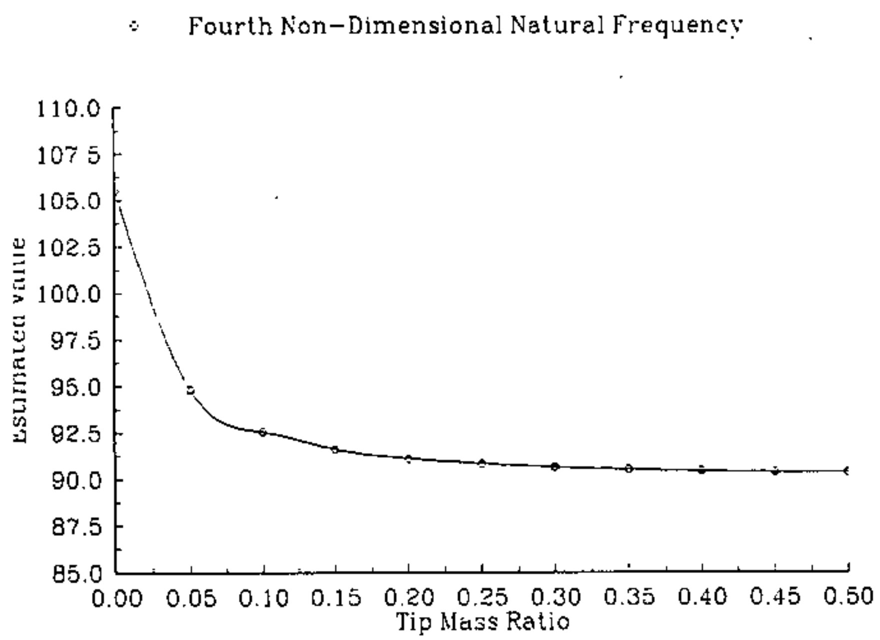
Fig(5.6) The effect of the tip mass ratio on the first non-dimensional natural frequency for fixed root exponentially varying cross section beam with η=1, θ=0, α=0, FORD=35, EPS= 1.0×10^{-6}



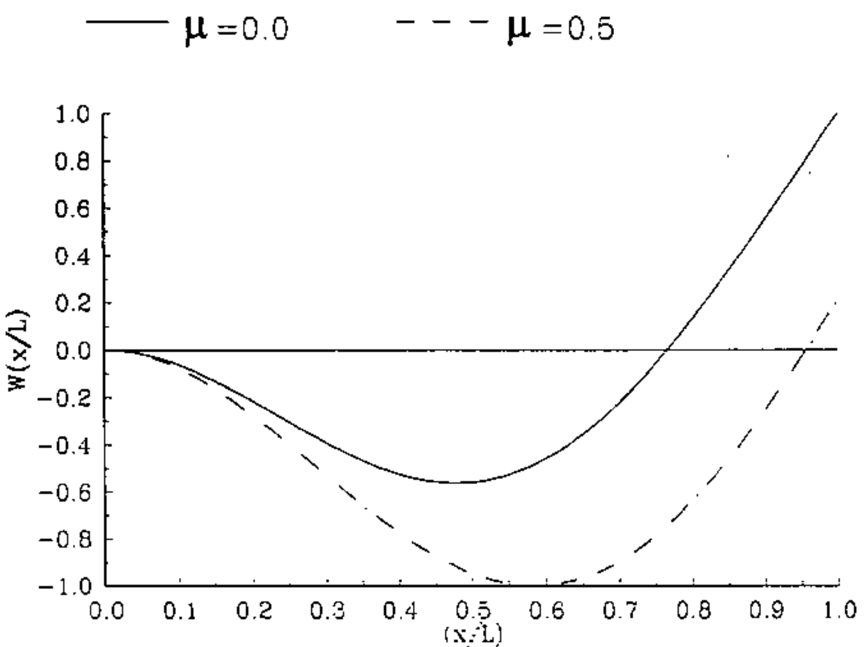
Fig(5.7) The effect of the tip mass ratio on the third non-dimensional natural frequency for fixed root exponentially varying cross section beam with η=1, θ=0, α=0, FORD=35, EPS= 1.0×10^{-6}



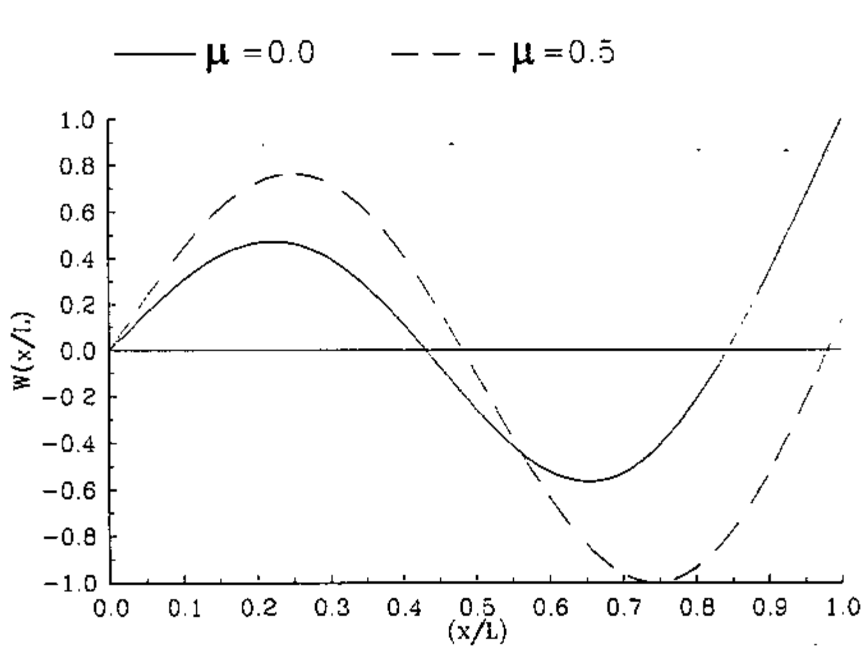
Fig(5.8) The effect of the tip mass ratio on the second non-dimensional natural frequency for hinged root exponentially varying cross section beam with $\eta=1$, $\theta=0$, $\alpha=0$, FORD=35, EPS= 1.0×10^{-6}



Fig(5.9) The effect of the tip mass ratio on the fourth non-dimensional natural frequency for hinged root exponentially varying cross section beam with $\eta=1$, $\theta=0$, $\alpha=0$, FORD=35, EPS= 1.0×10^{-6}



Fig(5.10) The effect of the tip mass ratio on the second mode shape for fixed root exponentially varying cross section beam with η=1, θ=0, α=0, FORD=35, EPS= 1.0×10^{-6}



Fig(5.11) The effect of the tip mass ratio on the third mode shape for fixedroot exponentially varying cross section beam with η=1, θ=0, α=0, FORD=35, EPS= 1.0×10^{-6}

non-dimensional natural frequencies. This is true regardless of the root attachment method and all other parameters, except the first non-dimensional natural frequency for hinged root case which is not affected by the presence of a tip mass since it is the rigid body mode of the motion.

Regarding the mode shapes fig(5.10) shows the second mode shape for a fixed root exponentially varying cross section properties beam with zero setting angle, zero root offset ratio and non dimensional rotational speed with tip mass ratio of zero and one half, using thirty five approximating function and tolerance 1.0×10^{-6}, also fig(5.11) shows the third mode shape for the same case but with hinged root. It is clear that the tip mass ratio significantly affects the mode shapes, actually this parameter is the one which has a considerable effect on the mode shapes. From the two figures shown, one notes that the tip mass always tends to increase the curvature of the mode shape and it also shifts the mode shape peak toward the root for fixed and hinged root cases, excepted from this the first mode shape of the hinged root case in which the mode shape is not affected by the tip mass ratio because it represents the rigid body mode of the motion.

5.4 The Effect of the Root Offset Ratio (α)

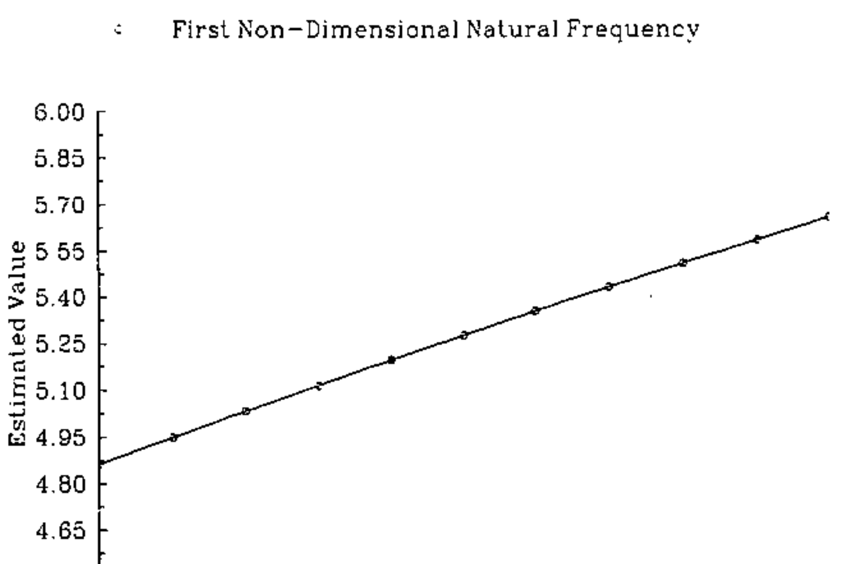
The root offset ratio is defined as the ratio of the hub radius (r) to the length of the beam (L). This parameter appears in the term related to the inertia force in the differential equation and in the boundary condition equations.

Non-Dimensional Natural Frequency $(\omega\sqrt{(\rho A_0 L^4)/(EI_0)})$				
	Series Method with FORD=35 and EPS=1.0×10^{-6}		Finite Elemnt Method With NFE=15 and EPS=1.0×10^{-6}	
No.	Fixed Root Case	Hinged Root Case	Fixed Root Case	Hinged Root Case
1	5.0321	1.6258	5.0298	1.6183
2	24.4677	16.8977	24.4691	16.8976
3	64.1295	51.4325	64.1771	51.4533
4	123.0008	105.6807	123.7221	105.9206

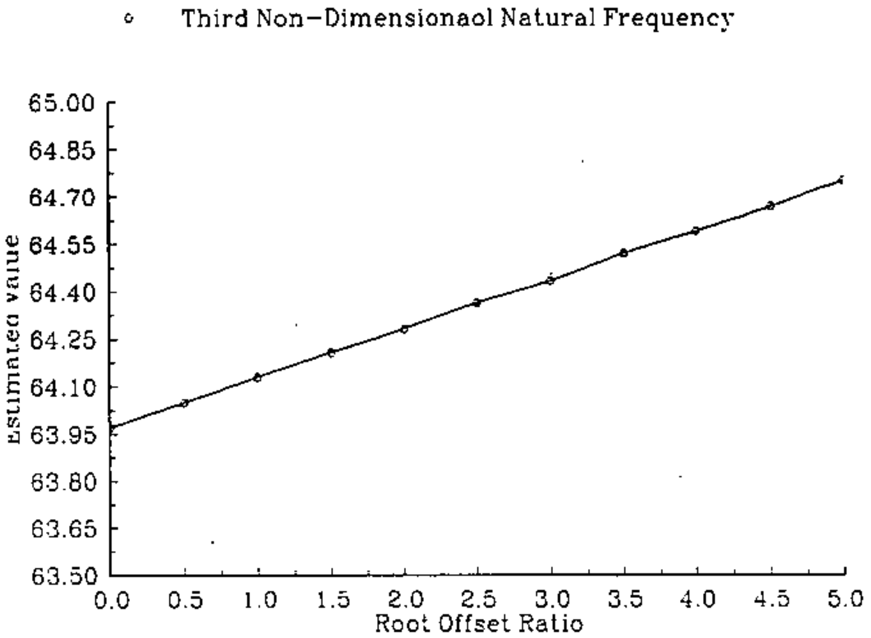
Table(5.4) Comparison of the results found by the series method with the finite element method results for an exponintially varying cross-section beam with $\eta=1$, $\mu=0$, $\theta=0$, $\alpha=1$

Table(5.4) shows a comparison of the results found from the finite element method with that found from the series method for exponentially varying cross-section properties beam with zero tip mass ratio, zero setting angle, root offset ratio equal one and non-dimensional rotational speed equal one for both fixed root and hinged root cases. The results show excellent agreement and supports the results found by the series method.

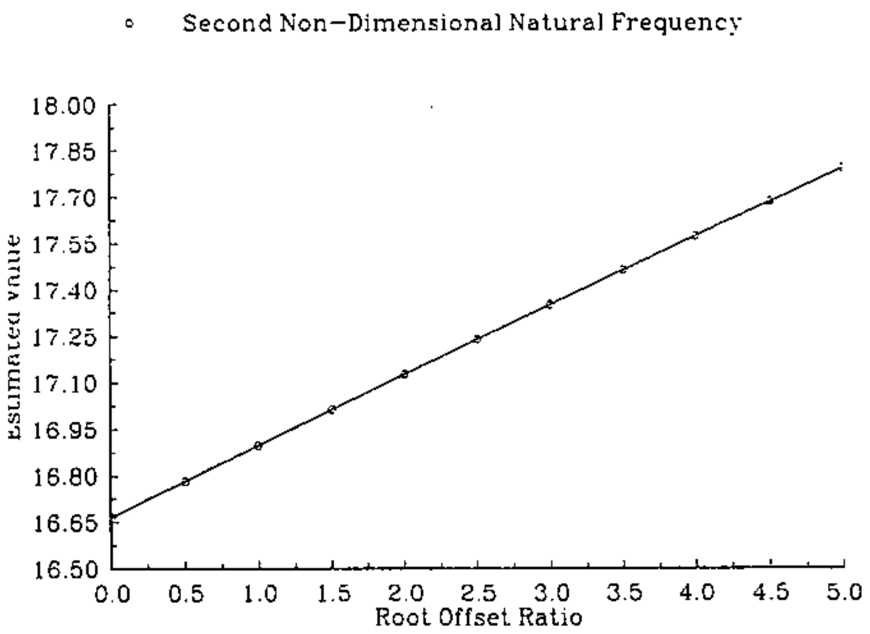
Figures (5.12) and (5.13) shows the effect of the root offset ratio on the first and third non-dimensional natural frequencies of a fixed root exponentially varying cross-section properties beam with zero tip mass ratio, zero setting angle and non-dimensional rotational speed equal one using thirty five approximating function order and tolerance EPS=1.0×10^{-6}, also figures (5.14) and (5.15) show the effect of the root offset ratio on the second and third non-dimensional natural frequencies for the above case with hinged root. The results obtained are shown in appendix C in table (C.5) and (C.6).



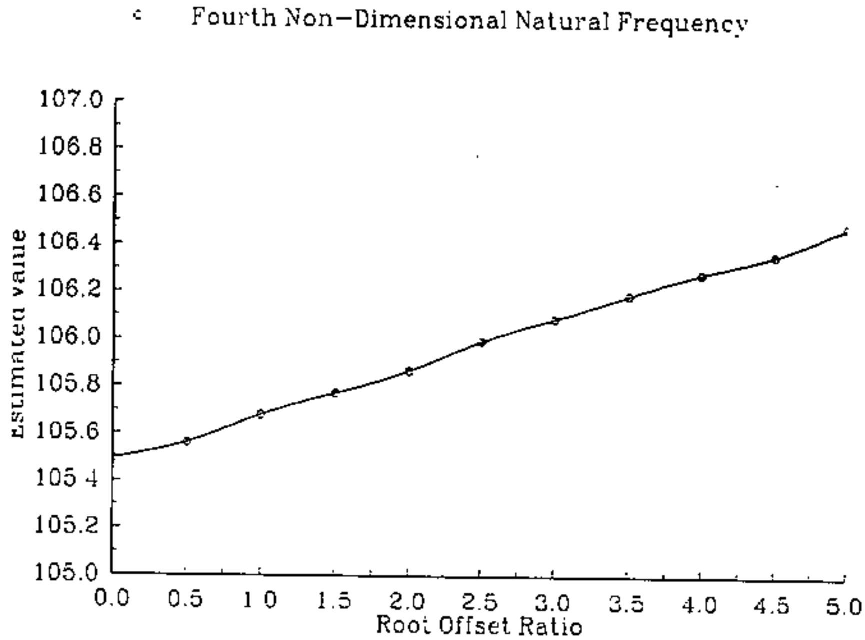
Fig(5.12) The effect of the root offset ratio on the first non-dimensional natural frequency for fixed root exponentially varying cross section beam with η=1, μ=0, θ=0, FORD=35, EPS= 1.0×10^{-6}



Fig(5.13) The effect of the root offset ratio on the third non-dimensional natural frequency for fixed root exponentially varying cross section properties beam with $\eta=1$, $\mu=0$, $\theta=0$, FORD=35, EPS= 1.0×10^{-6}



Fig(5.14) The effect of the root offset ratio on the second non-dimensional natural frequency for hinged root exponentially varying cross section properties beam with $\eta=1$, $\mu=0$, $\theta=0$, FORD=35, EPS= 1.0×10^{-6}



Fig(5.15) The effect of the root offset ratio on the fourth non-dimensional natural frequency for hinged root exponentially varying cross section properties beam with $\eta=1$, $\mu=0$, $\theta=0$, FORD=35, EPS= 1.0×10^{-6}

From these figures and the results shown in appendix C it is noted that increasing the root offset ratio will increase the estimated non-dimensional natural frequency in both fixed root and hinged root cases, also it is noted that for the low non-dimensional natural frequencies the effect is more significant. It is also clear that the variation of the non-dimensional natural frequencies with the root offset ratio is linear.

The root offset ratio hasn't any effect on the mode shapes, for many different values of the root offset ratio and different cases the mode shapes were drawn and in all cases the mode shapes showed no difference.

5.5 The Effect of the Setting Angle (θ)

The setting angle is defined to be the angle between the direction of rotation and the direction of vibration which is the same as that between the normal to the direction of rotation and the normal to the direction of vibration, the setting angle appears in the terms related to the kinetic energy appears with the non-dimensional natural frequency so it is expected that the setting angle will affect the non-dimensional natural frequencies only and hasn't any effect on the mode shapes.

Table(5.5) shows a comparison of the results found by the series method with the finite element method, for an exponentially varying cross-section properties beam with zero tip mass ratio, zero root offset ratio, non-dimensional rotational speed equal one, using setting angle of forty five degrees for both fixed root and hinged root cases. From this table it is clear that the results show excellent agreement.

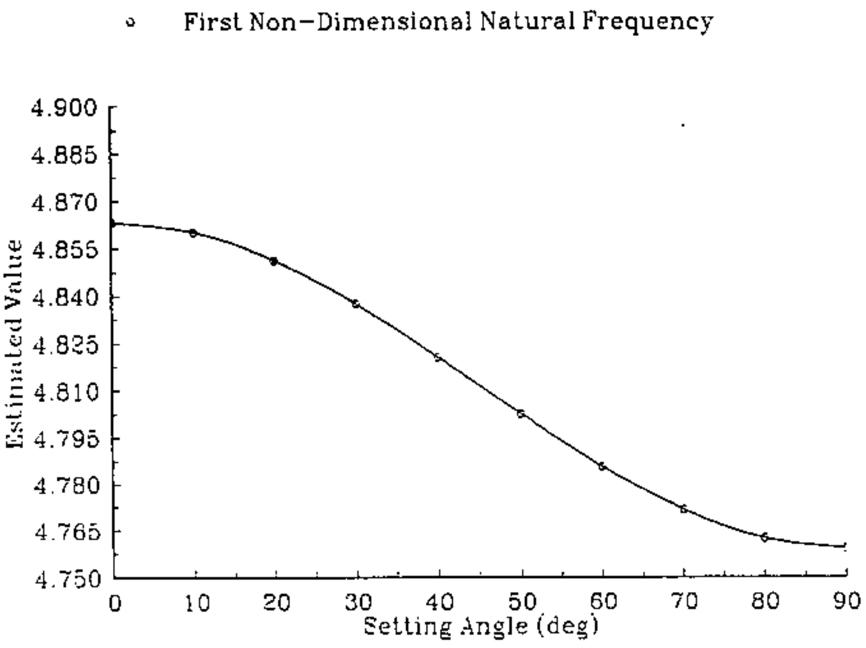
Figures (5.16) and (5.17) show the effect of the setting angle on the first and third non-dimensional natural frequencies of a fixed root, exponentially varying cross-section properties beam with zero tip mass ratio, zero root offset ratio and non-dimensional rotational speed equal one, also figures (5.18) and (519) show the effect of the setting angle on the second and fourth non-dimensional natural frequencies for the same case but with hinged root, similarly figures (5.20) and (5.21) show the effect of the setting angle on the first and second non-dimensional natural frequencies for the fixed root case with the non-dimensional rotational speed equal five. From these figures it is noted that the increase in the setting angle will reduce the estimated non-dimensional natural frequency regardless of the case under consideration. For small and high setting angles the change in the setting angle has smaller effect on the non-dimensional natural frequencies relative to the intermediate values of the setting angle, also the change in the non-dimensional natural frequency due to the change in the setting angle is similar to the sin function and this is expected because the effect of the setting angle appears in the form of the square of the sin function. The magnitude of the effect of the setting angle on any non-dimensional natural frequency depends on two factors, the first is the non-dimensional natural frequency itself and the second is the non dimentional rotational speed. If the non-dimensional natural frequency under consideration is small then the effect of the setting angle is greater than the effect for high non-dimensional natural frequencies; this because the non-dimensional natural frequencies are calculated from :-

$$\sqrt{\frac{\rho A_0 L^4}{EI_0}}\omega = \sqrt{\lambda - \eta^2 \sin^2(\theta)} \qquad \text{.............. (4.4)}$$

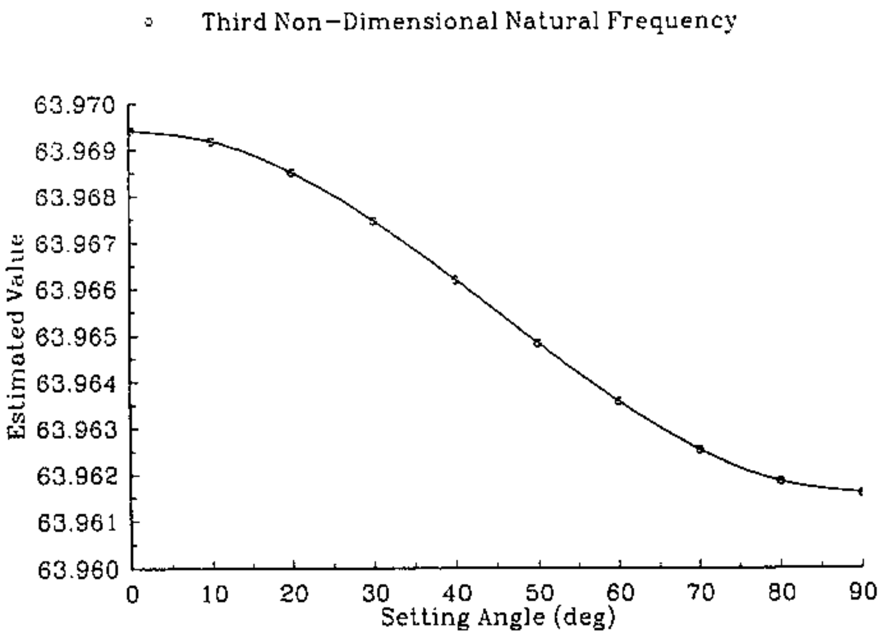
all the figures shown in this section support this expectation,. Also since the square of the product of the sin of the sitting angle and the non-dimensional rotational speed is subtracted from the eigenvalue, increasing the non-dimensional rotational speed is expected to reduce the estimated non-dimensional natural frequency more and more. Comparing figures (5.16), (5.17), (5,20) and (5.21) which represents the effect of the setting angle on the same non-dimensional natural frequencies but with different non-dimensional rotational speed. It is clear that these curves reflects exactly what has been expected.

Non-Dimensional Natural Frequency $(\omega\sqrt{(\rho A_0 L^4)/(EI_0)})$				
Series Method with FORD=35 and EPS=1.0×10^{-6}			Finite Elemnt Method With NFE=15 and EPS=1.0×10^{-6}	
No.	Fixed Root Case	Hinged Root Case	Fixed Root Case	Hinged Root Case
1	4.8115	0.7071	4.8109	0.7115
2	24.3037	16.6510	24.3033	16.6562
3	63.9655	51.2283	63.9756	51.2358
4	122.7009	105.4914	123.2734	105.5299

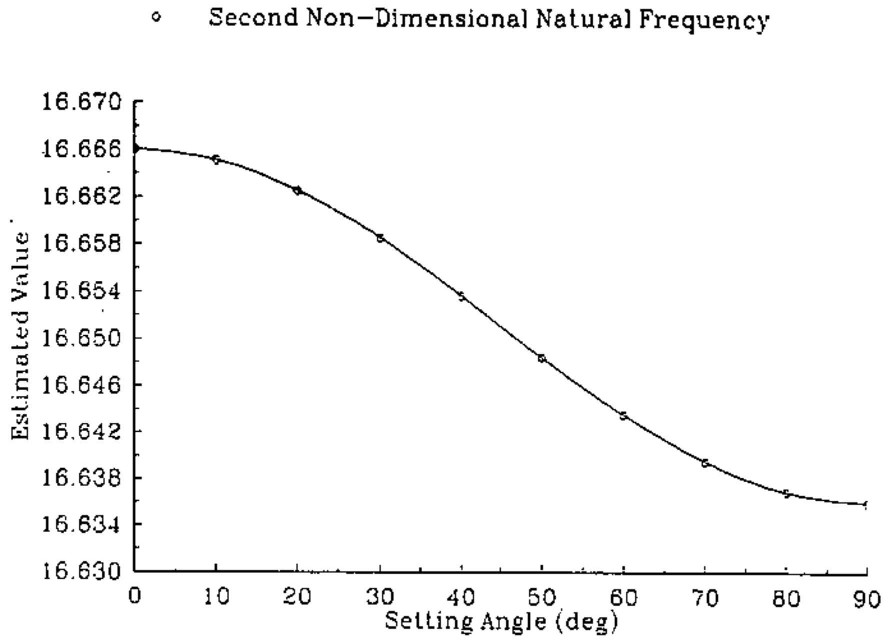
Table(5.5) Comparison of the results obtained by the series method with the finite element method results for an exponentially varing cross-section beam with $\eta=1$, $\mu=0$, $\theta=45$, $\alpha=1$



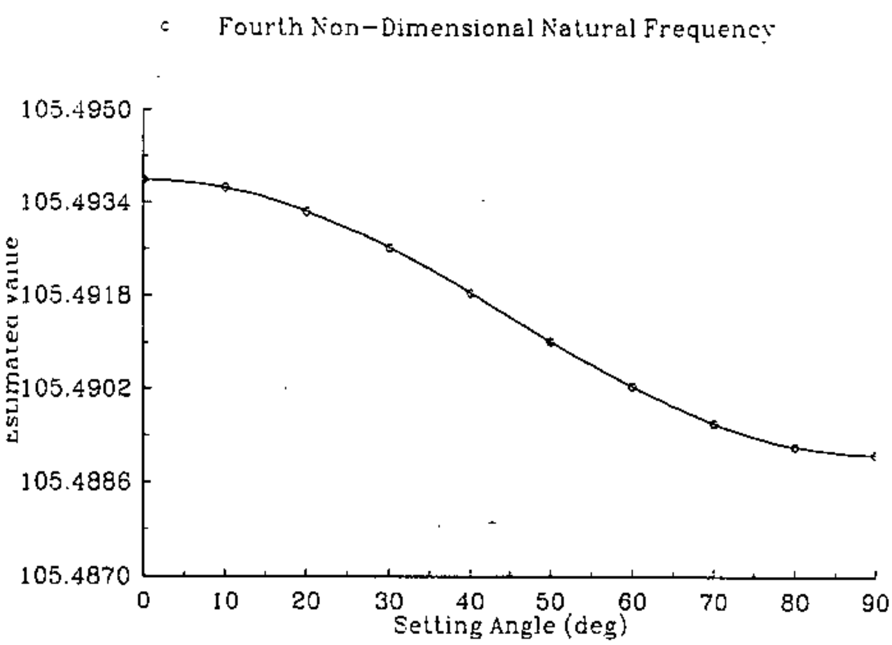
Fig(5.16) The effect of the setting angle on the first non-dimensional natural frequency for fixed root exponentially varying cross section beam with $\eta=1$, $\mu=0$, $\alpha=0$, FORD=35, EPS= 1.0×10^{-6}



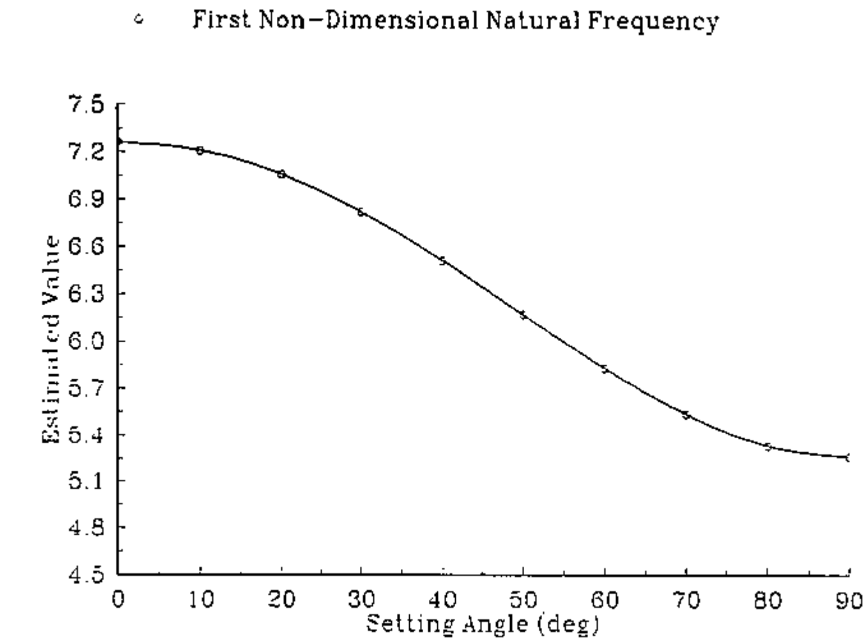
Fig(5.17) The effect of the setting angle on the third non-dimensional natural frequency for fixed root exponentially varying cross section beam with $\eta=1$, $\mu=0$, $\alpha=0$, FORD=35, EPS= 1.0×10^{-6}



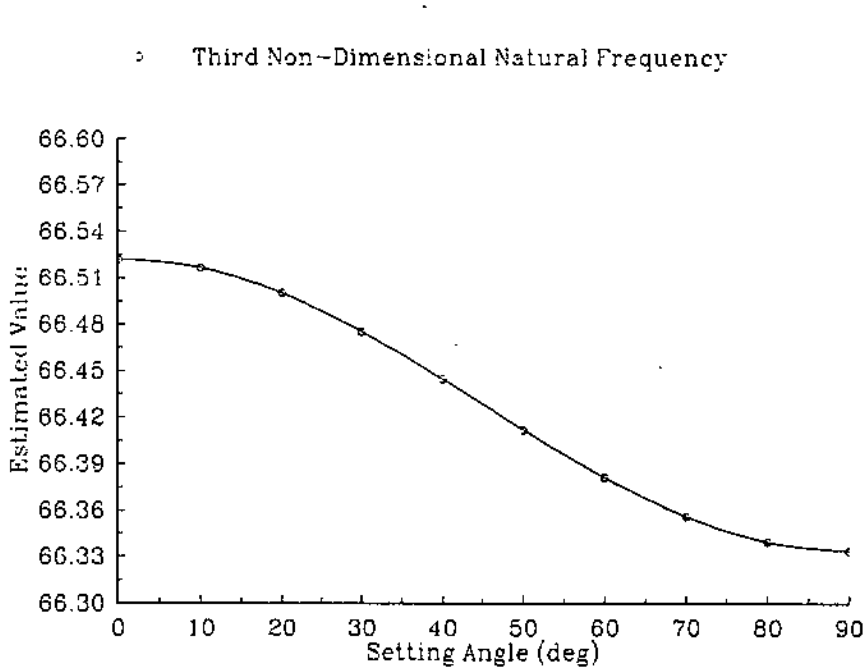
Fig(5.18) The effect of the setting angle on the second non-dimensional natural frequency for inged root exponentially varying cross section beam with η=1, μ=0, α=0, FORD=35, EPS= 1.0×10^{-6}



Fig(5.19) The effect of the setting angle on the fourth non-dimensional natural frequency for hinged root exponentially varying cross section beam with η=1, μ=0, α=0, FORD=35, EPS= 1.0×10^{-6}



Fig(5.20) The effect of the setting angle on the first non-dimensional natural frequency for fixed root exponentially varying cross section beam with η=5, μ=0, α=0, FORD=35, EPS= 1.0×10^{-6}



Fig(5.21) The effect of the setting angle on the third non-dimensional natural frequency for fixed root exponentially varying cross section beam with η=5, μ=0, α=0, FORD=35, EPS= 1.0×10^{-6}

The setting angle was found to have no effect on the mode shapes and this is expected because the setting angle is always related to the eigen values found and hasn't any relation with the mode shapes

5.6 The Effect of the Translation Stiffness Ratio (β)

The translation stiffness ratio is defined as the ratio of the translation stiffness to the flexural rigidity at the root times the length to the third power. In most of the cases the hub to which the rotating beam is attached is very stiff so it can be considered rigid but this doesn't elemenat the possibility of having translation flexibility at the root and sometimes the effect of this flexibility at the root can not be neglected.

To include the two limiting cases of zero translation stiffness ratio and the infinite case in which the root can be considered fixed, a very high rotational stiffness ratiois is used. By studying the effect of rotational stiffness ratio it was found that taking the rotational stiffness ratio to be of the order 1.0×10^{10} , the beam can be considered rotationally riged, so in studying the effect of the translation stiffness ratio the rotational stiffness ratio is taken to be 10×10^{10}

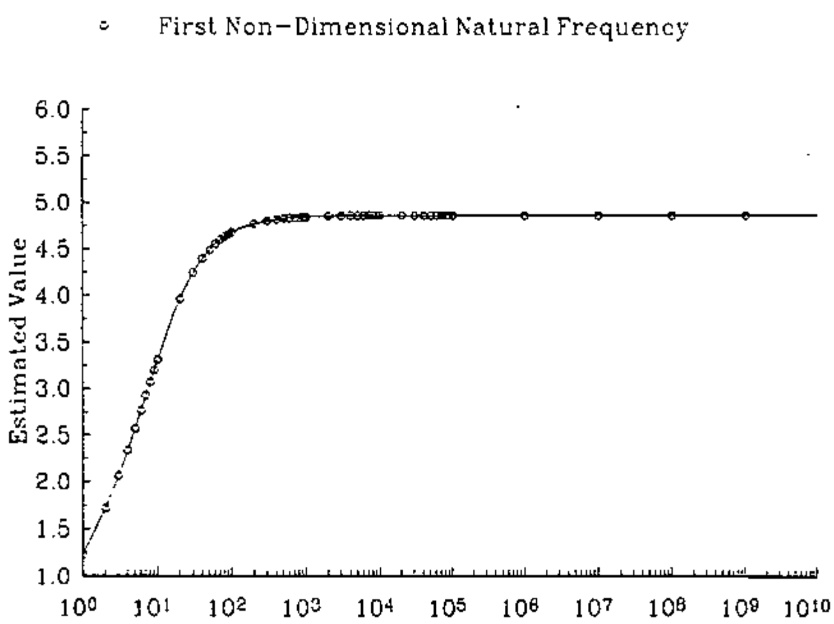
Table(5.6) shows a comparison for the results found by the finite element method with that found from the series method for exponentially varying cross-

Non-Dimensional Natural Frequency $(\omega\sqrt{(\rho A_0L^4)/(EI_0)})$				
	Series Method with FORD=35 and EPS=1.0×10^{-6}		Finite Elemnt Method With NFE=15 and EPS=1.0×10^{-6}	
No.	β=100	β=500	β=100	β=500
1	4.6764	4.8262	4.6767	4.8267
2	16.0951	22.3095	16.0947	22.3093
3	35.2100	48.9693	35.2107	48.4984
4	77.3670	84.2597	77.3735	84.2693

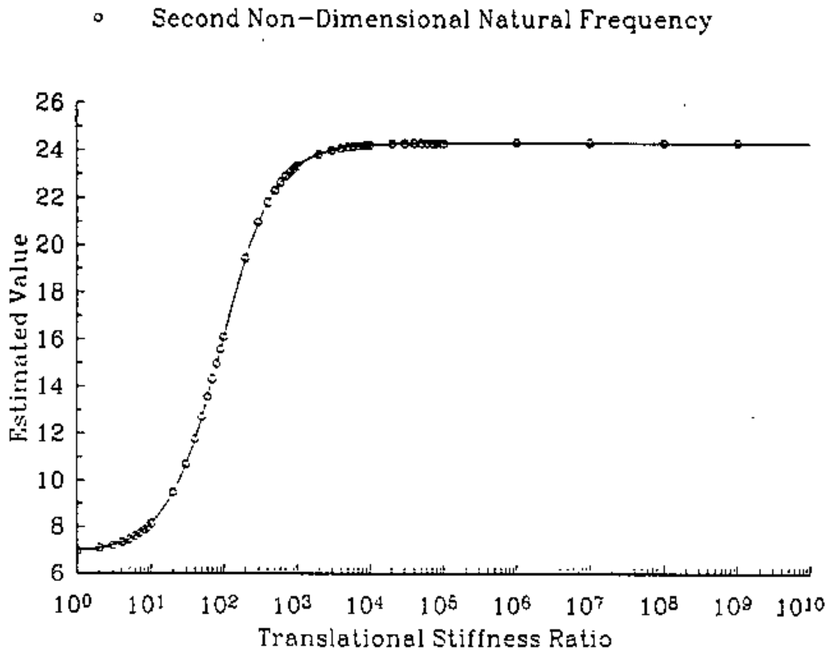
Table(5.6) Comparison of the results obtained by the series method with the finite element method results. for exponential varying cross-section beam with η=1, μ=0, θ=0, α=0, γ= 1.0×10^{11}

section- properties variation with zero tip mass ratio, zero setting angle, zero root offset ratio and non-dimensional rotational speed is equal one. For this case the translation stiffness ratio is equal 100 in one case and 500 in the second case, the rotational stiffness ratio is 10×10^{10}. As shown in this table the results show excellent agreement and the error is very small.

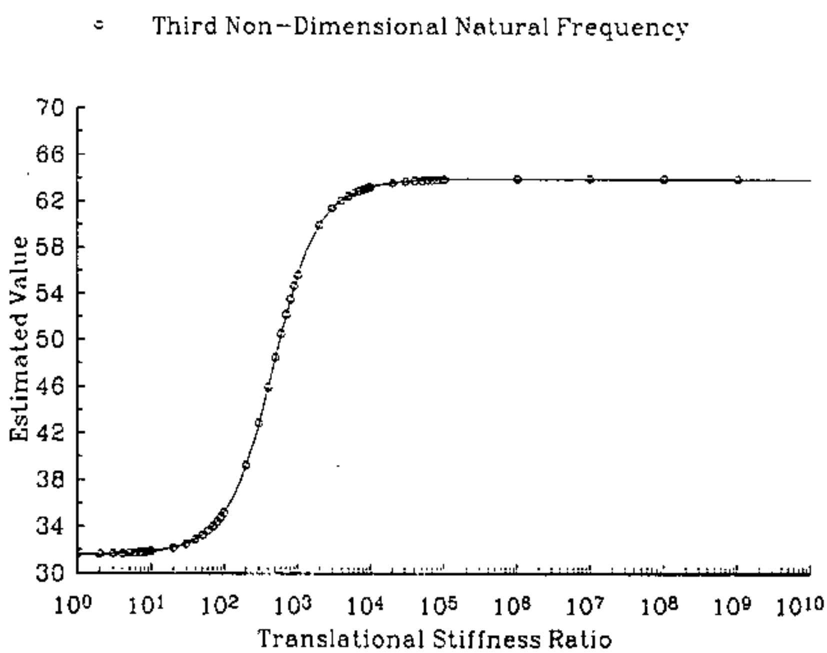
From the results presented in appendix C and the figures (5.22), (5.23), (5.24) and (5.25) which show the variation in the first four non-dimensional natural frequencies it is noted that the translation stiffness ratio always increases the non-dimensional natural frequency, but there is always a range in which the translation stiffness ratio has great effect in increasing the non-dimensional natural frequency and this range depends on the non-dimensional natural frequency itself, for example the first non-dimensional natural frequency has the range of about zero to about one hundred, the most effective range of the translation stiffness ratio on the second non-dimensional natural frequency is about 10 to 1000, for the third non-



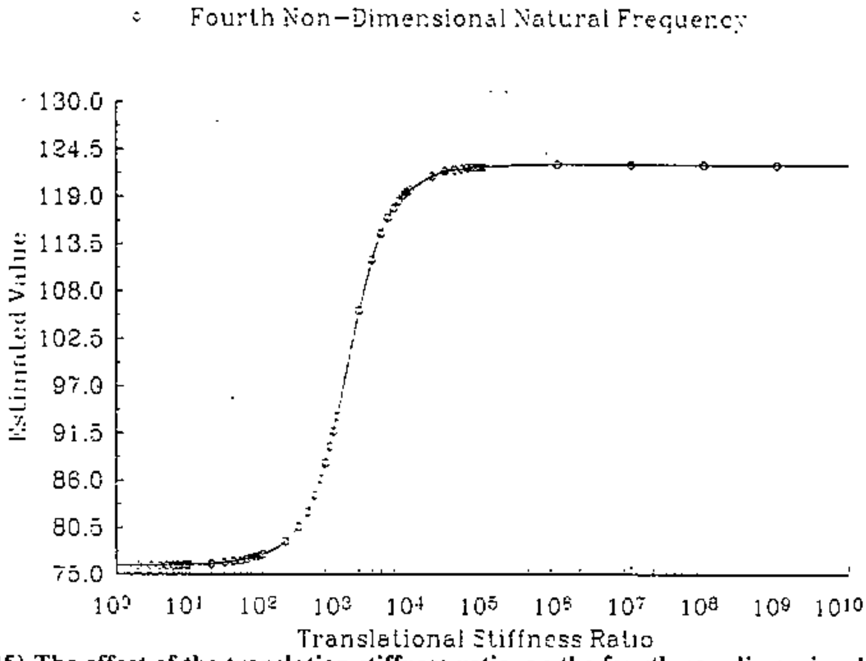
Fig(5.22) The effect of the translation stiffness ratio on the first non-dimensional natural frequency for exponentially varying cross section beam with $\eta=1$, $\mu=0$, $\theta=0$, $\alpha=0$, $\gamma=1.0\times10^{11}$, FORD=35, EPS= 1.0×10^{-6}



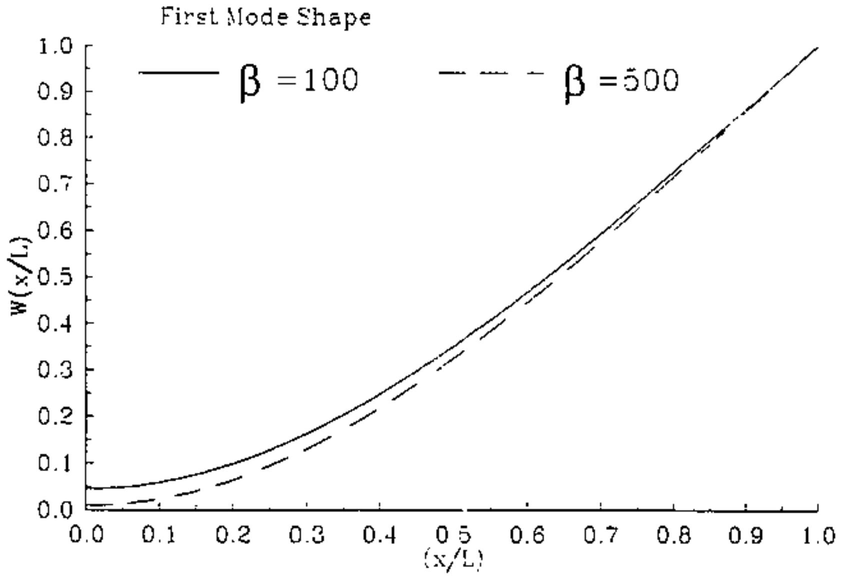
Fig(5.23) The effect of the translation stiffness ratio on the second non-dimensional natural frequency for exponentially varying cross section beam with η=1, μ=0, θ=0, α=0, γ= 1.0×10^{11}, FORD=35, EPS= 1.0×10^{-6}



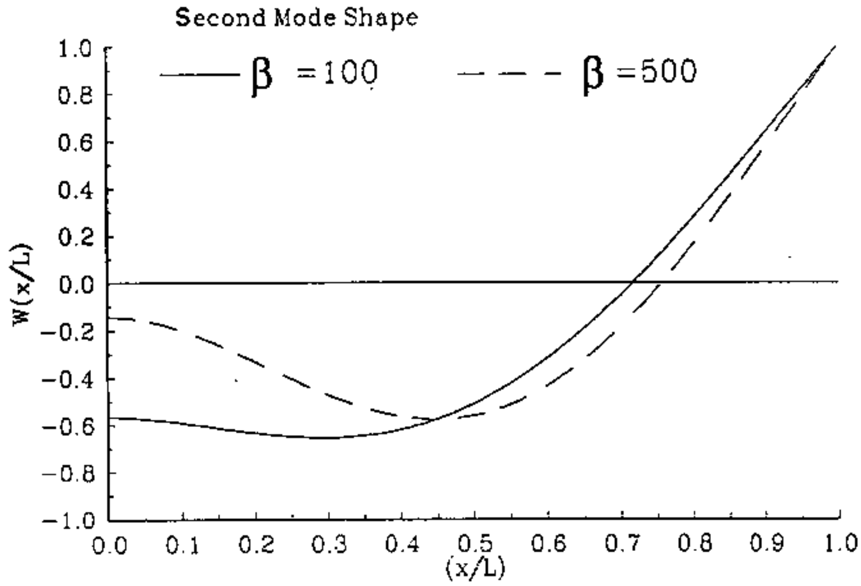
Fig(5.24) The effect of the translation stiffness ratio on the third non-dimensional natural frequency for exponentially varying cross section beam with η=1, μ=0, θ=0, α=0, γ= 1.0×10^{11}, FORD=35, EPS= 1.0×10^{-6}



Fig(5.25) The effect of the translation stiffness ratio on the fourth non-dimensional natural frequency for exponentially varying cross section beam with η=1, μ=0, θ=0, α=0, γ= 1.0×10^{11}, FORD=35, EPS= 1.0×10^{-6}



Fig(5.26) The effect of the translation stiffness ratio on the first mode shape for exponentially varying cross section properties beam with η=1, μ=0, θ=0, α=0, γ= 1.0×10^{11}, FORD=35, EPS= 1.0×10^{-6}



Fig(5.27) The effect of the translation stiffness ratio on the second mode shape for exponentially varying cross section properties beam with η=1, μ=0, θ=0, α=0, γ= 1.0×10^{11}, FORD=35, EPS= 1.0×10^{-6}

dimensional natural frequency the range is 100 to 5000, for the fourth non-dimensional natural frequency the effective range is 100 to 10000, and in this range the non-dimensional natural frequency changes from the non-dimensional natural frequency with zero translation stiffness ratio to the limiting case of infinity translation stiffness ratio. Outside of these ranges the curves are approximately flat. Also the width of these ranges increase with increasing the non-dimensional natural frequency number, in the same way the range over which the non-dimensional natural frequency vary increase with increasing the non-dimensional natural frequency number.

The translation stiffness ratio is expected to have great effect on the mode shapes. As it is seen from figures (5.26) and (5.27), the mode shapes will have a deflection at the root, this is theoretically expected. The deflection at the root will be high for low translation stiffness ratio, this is shown in the figures and it is expected, also from these figures it is noted that the slope at the root is zero this because a very high rotational stiffness ratio has been used.

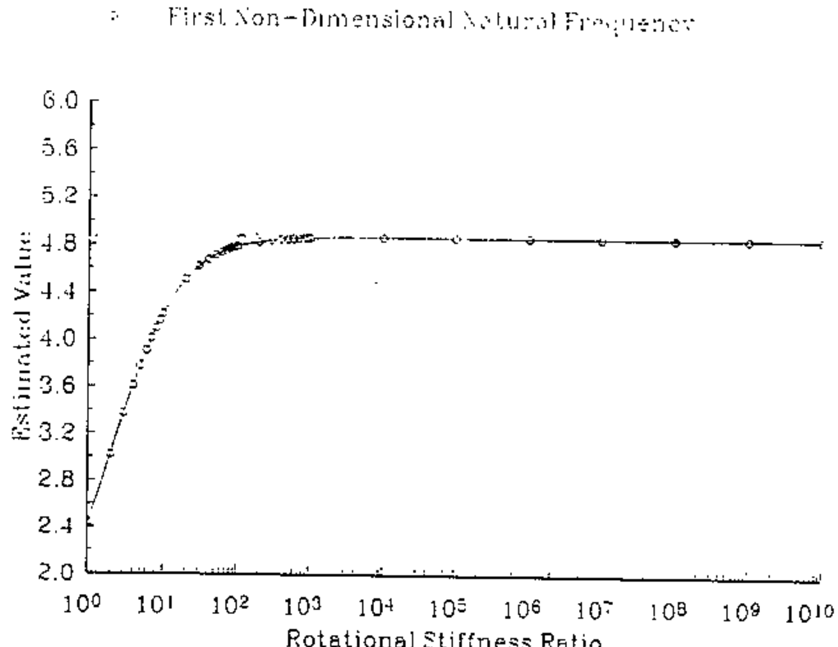
5.7 The effect of the Rotational Stiffness Ratio (γ)

The rotational stiffness ratio as defined in section(2.3) is the ratio of the rotational stiffness to the flexural rigidity at the root times the length. This parameter is expected to affect the non-dimensional natural frequencies and the mode shapes. For the mode shapes, the slope of the mode shapes at the root will vary by varying the rotational stiffness ratio. In this study a very high value of the translation stiffness ratio (β) is used, from the previous section the non-dimensional translation stiffness ratio (β) is taken to be 10×10^{10}.

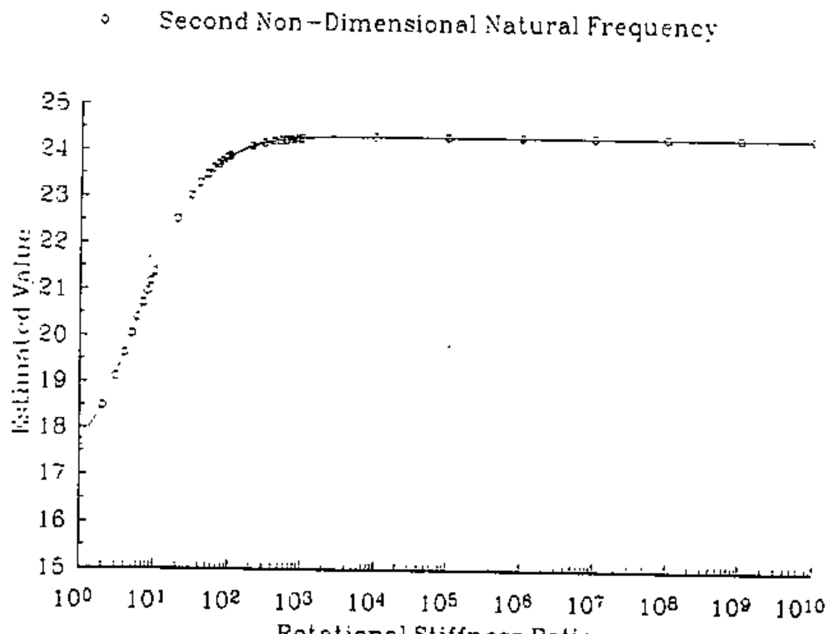
Table(5.7) shows a comparison of the first four non-dimensional natural frequencies for the results found from the finite element method with that found from the series method. These results are found for an exponentially varying cross-section properties beam with zero tip mass ratio, zero setting angle, zero root offset ratio, non-dimensional rotational speed equal one and translation stiffness ratio 10×10^{10} using rotational stiffness ratio of 100 for the first case and 500 for the second case. From this table it is clear that the results show excellent agreement.

Non-Dimensional Natural Frequency $(\omega\sqrt{(\rho A_0 L^4)/(EI_0)})$				
	Series Method with FORD=35 and EPS=1.0×10^{-6}		Finite Elemnt Method With NFE=15 and EPS=1.0×10^{-6}	
No.	γ=100	γ=500	γ=100	γ=500
1	4.7846	4.8472	4.7843	4.8484
2	23.8773	24.2229	23.8774	24.2229
3	62.8213	63.7299	62.8229	63.7299
4	120.6590	122.3047	121.0539	122.7567

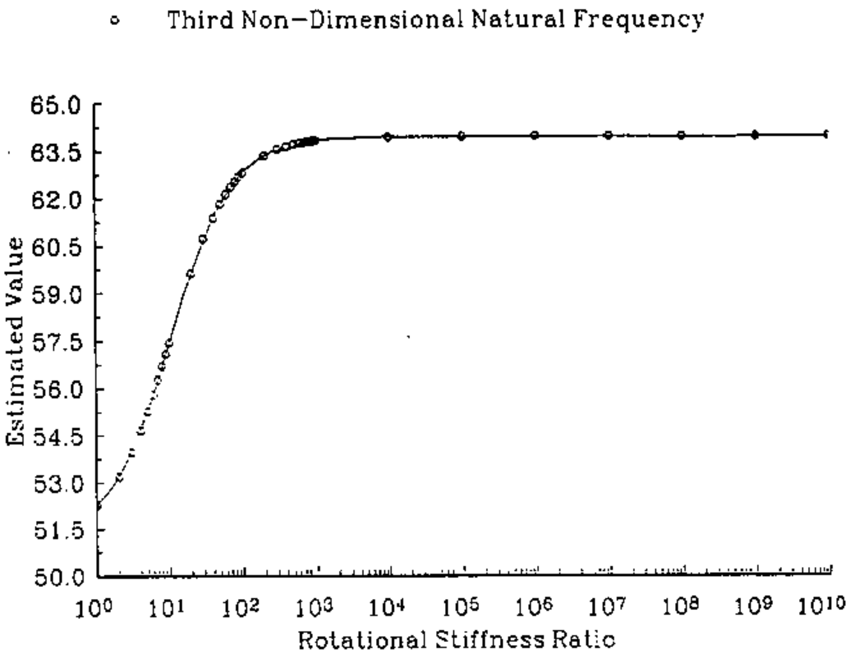
Table(5.7) Comparison of the result obtained fro the finite element methods with the series method results for an exponential varying cross-section beam with η=1, μ=0, θ=0, α=0, β= 1.0×10^{11}



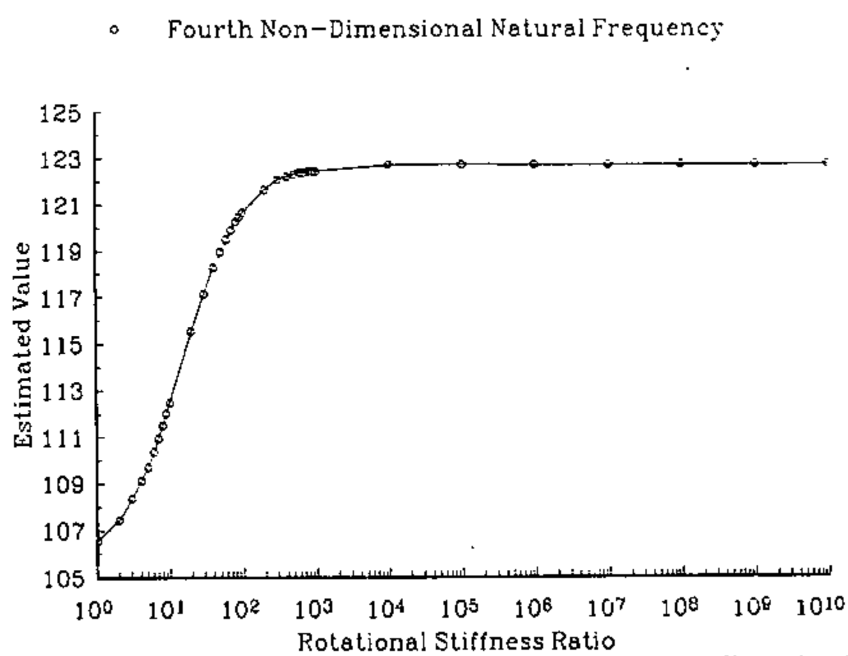
Fig(5.28) The effect of the rotational stiffness ratio on the first non-dimensional natural frequency for exponentially varying cross section beam with η=1, μ=0, θ=0, α=0, β= 1.0×10^{11}, FORD=35, EPS= 1.0×10^{-6}



Fig(5.29) The effect of the rotational stiffness ratio on the second non-dimensional natural frequency for exponentially varying cross section beam with η=1, μ=0, θ=0, α=0, β= 1.0×10^{11}, FORD=35, EPS= 1.0×10^{-6}

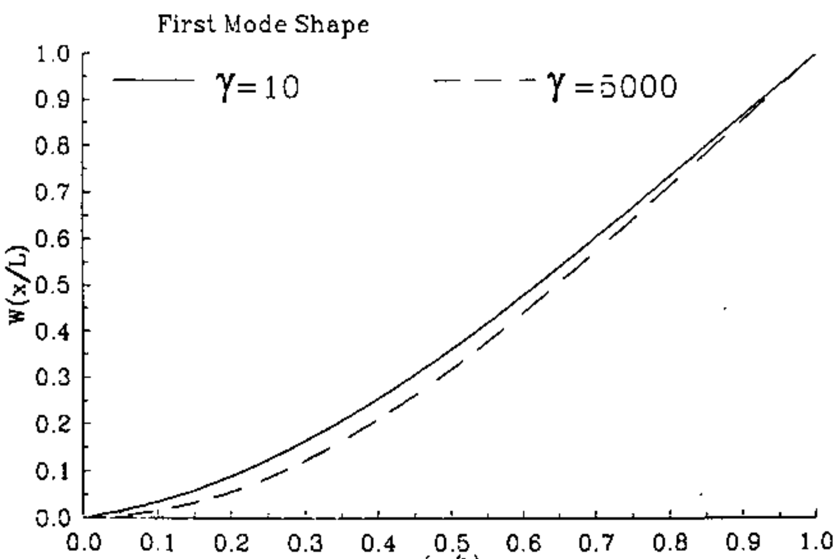


Fig(5.30) The effect of the rotational stiffness ratio on the third non-dimensional natural frequency for exponentially varying cross section beam with $\eta=1$, $\mu=0$, $\theta=0$, $\alpha=0$, $\beta= 1.0\times10^{11}$, FORD=35, EPS= 1.0×10^{-6}

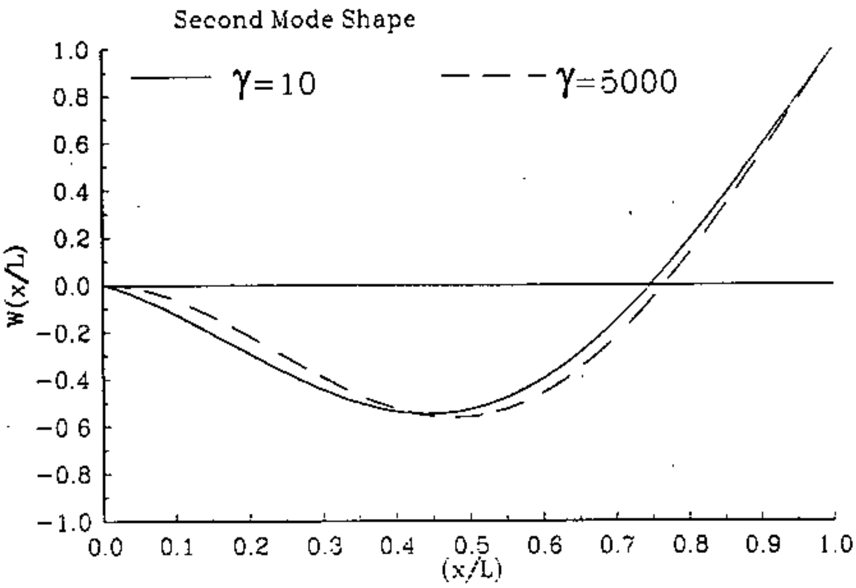


Fig(5.32) The effect of the rotational stiffness ratio on the fourth non-dimensional natural frequency for exponentially varying cross section beam with $\eta=1$, $\mu=0$, $\theta=0$, $\alpha=0$, $\beta= 1.0\times10^{11}$, FORD=35, EPS= 1.0×10^{-6}

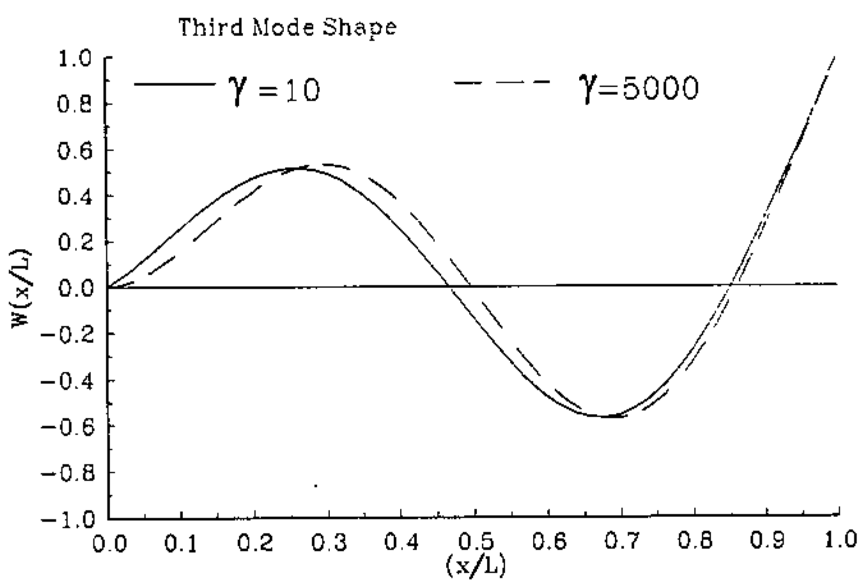
Figures (5.28), (5.29), (5.30), (5.31) and the numerical results shown in appendix C show a study for the effect of the rotational stiffness ratio. From these results it is noted that increasing the rotational stiffness ratio always increases the non-dimensional natural frequency. The effect of the rotational stiffness ratio is small compared to the effect of the translation stiffens ratio especially for high non-dimensional natural frequencies. Similar to the effect of the translation stiffness ratio, there is a range in which the change in the rotational stiffness ratio is dominant and outside that range the non-dimensional natural frequency curves appears flat. These ranges and their width depend on the non-dimensional natural frequency under consideration, but all of them starts from zero rotational stiffness ratio. For example the effective range of the rotational stiffness ratio on the first non-dimensional natural frequency is in the range of zero to one hundred. For the second one form zero to one hundred fifty, for the third non-dimensional natural frequency the range is zero to about two hundreds and for the fourth the range starts by zero and ends with about five hundreds.



Fig(5.32) The effect of the rotational stiffness ratio on the first mode shape for exponentially varying cross-section beam with $\eta=1$, $\mu=0$, $\theta=0$, $\alpha=0$, $\beta= 1.0\times10^{11}$, FORD=35, EPS= 1.0×10^{-6}



Fig(5.33) The effect of the rotational stiffness ratio on the second mode shape for exponentially varying cross-section beam with $\eta=1$, $\mu=0$, $\theta=0$, $\alpha=0$, $\beta= 1.0\times10^{11}$, FORD=35, EPS= 1.0×10^{-6}



Fig(5.34) The effect of the rotational stiffness ratio on the third mode shape for exponentially varying cross-section beam with $\eta=1$, $\mu=0$, $\theta=0$, $\alpha=0$, $\beta= 1.0\times10^{11}$, FORD=35, EPS= 1.0×10^{-6}

The rotational stiffness ratio affects the mode shapes such that the increase in the rotational stiffness ratio will decrease the slop of the mode shape curve at the root, and by this the mode shape is changed so that for the same boundary condition at the tip the curve will change until it nearly retain the same slop at the tip. This is also clear from figures (5.32), (5.33) and (5.34) which show a comparison for the first three mode shapes for the considered case with rotational stiffness ratio of 10 and 5000. This is true for all mode shapes regardless of the other parameters and boundary conditions.

Chapter Six

Results, Conclusions and Recommendations

From this work many results were obtained concerning the finite element method, the power series method and the parameters affecting the problem of free vibration of rotating beams with non-uniform cross-section. These results were discussed throughout this work . They can be summarized in the following:-

1- The results found by the finite element method are sufficiently accurate so it is acceptable as a method to compare results from other newly developed methods.

2- The finite element method solution is an approximated solution . Its accuracy depends on the number of finite elements used, the solution convergence tolerance and the case considered. For any considered case there are an optimal values for the number of finite elements and the tolerance for which the error in the results obtained is minimum. These values should be found for each case considered.

3- The running time of the finite element computer program depends on the number of finite elements, tolerance and the number of non-dimensional natural frequencies required in addition to its dependence on the case considered and the speed of the computer used. The running time varies from a few minuets to more than half an hour depending on the factors stated above.

4- The power series method is a very powerful method. It is the only analytical method that can solve differential equations with variable coefficients exactly. It suffers from the truncation errors.

5- The factors that affect the error in the power series solution are the approximating function order, the solution convergence tolerance and the cross-section properties variation series order.

6- The running time of the series method computer program depends on the problem considered, the approximating function order, the solution convergence tolerance and the speed of the computer used. Depending on these factors the running time varies from less than one minuet to more than half an hour

7-The increase in the non-dimensional rotational speed tends to increase the non-dimensional natural frequencies. Its increase also reduces the curvature of the mode shapes slightly.

8- The increase in the tip mass ratio decreases the non-dimensional natural frequencies. It also affects the mode shapes by forcing the peak of the mode shape to move towards the root.

9- The root offset ratio increase always increases the non-dimensional natural frequencies, but it doesn't change the mode shapes.

10- The setting angle increase reduces the non-dimensional natural frequencies but it doesn't affect the mode shapes at all.

11- The increase in the translation stiffness ratio affects both the non-dimensional natural frequencies and the mode shapes. Its increase increases the non-dimension natural frequencies , at the same time its presence causes a deflection at the root which decreases with increasing the translation stiffness ratio.

Finally it is recommended that :-

1- The results presented in this work are exact, so it could be used in design of rotating parts which can be modeled as a rotating beam. Also these results can be used to compare the results of other methods.

2-The problem of free vibration of rotating beams with non-uniform cross-section that are modelled as Timoshinko beams can be handled by the power series technique developed in this thesis.

3- The power series method can be used to solve any problem governed by a differential equation with variable coefficients, the only condition on the application of this method is that the coefficients are written in a power series form and all the points within the domain of the differential equation are either ordinary or regular singular points[1].

[1] REF[16]. REF[19] and REF[22].

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Appendix A

The Finite Element Method

A.1 The Finite Element Procedure

The finite element method is a numerecal procedure for solving physical problems governed by a differential equation or an energy theorem. It has two main characterestics which distinguish it from other methods :-

1- The method utilize an integral formulation to get a system of algepriac equations.

2- This method uses continuose, picewise smooth functions for approximating the unknown quantity or quantities.

The advanteges of the finite elemnt method is clear. it is easilly applied to irregular shaped objects composed of several different materials and having mixed boundary conditions. it is applied to steady-state and time dependant problems as well as problems involving nonlinear material properties, because of its simplicity and power it is widly used in heat transfer, fluid mechanics and structural analysis in addition to its use in solving differential equations.

The finite element method involves six main steps which should be conducted throughout the solution of any problem, these steps are :-

1- Discritization of the region or domain of the problem

This step includes the spesification of the finite element shape, the generation of the finite element mesh and numbering the nodal points as well as specifing the coordinates values.

2- Specifying the approximating equation (interpolation function)

The approximating function may take any form and has no restriction excipt it being continuose and is derevatives are defined up to the maximum required one. It was found that the polynomial type is one of the best types that could be used in the finite element method, the polynomial used is always of the form :-

$$W(x) = \sum_{i=1}^{4} C_i x^i \qquad \text{.............. (A.1)}$$

The approximating function we used in this work was of this type i.e. polynomial type. This equation should be written in terms of the nodal values, the order (n) of the approximationg polynomial depends on the number of the degrees of freedom for the element (nodal values per element).

3- The development of the element equation.

in this step either the variational method or the weighted residual method is used, then the variational integral is minimized with respect to the nodal values to get the element equation in the form :-

$$[a]\{W\} = \{F\} \qquad \text{.............. (A.2)}$$

4- Assemply of the element equations to obtaine the equation of the whole system.

5- Imposing of the boundary conditions of the problem.

6- Solution of the system equation for the unknown quantity or quantities

In this step we first solve for the nodal values, then estimate the required quantity or quantities and represnts them in a sutable form.

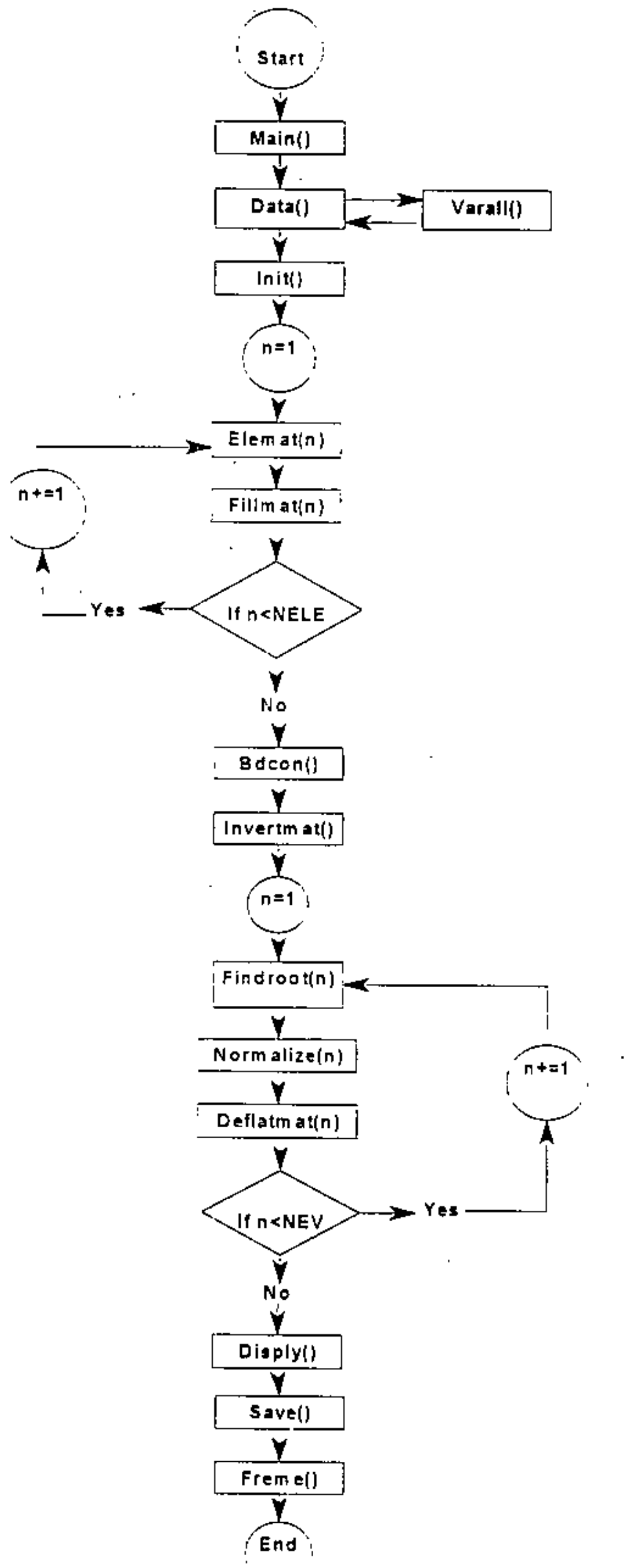
The procedure was easy to be applied because it was systematic and flexible; it was easy to be programmed and to be applied to any type of problems and which made it the most sued in engineering.

A.2 The Solution of The problem of Vibration of Rotating Beams With Non-Uniform Cross-Section For Which the Cross-Section Variations are Given in Series Form Using The Finite Element Method (Computer Program)

The computer program shown bellow was written in C Language to solve the problem of free vibration of rotating beams with non-uniform cross section for which the cross section properties variation ids given in series power form. The stiffness and mass matrices were developed in chapter three with the boundary conditions Figure (A.1) show a flow chart for this computer program.

Since the C language is a structural language, the program was divided into functions. These functions are divided into two sets, the first which estimate and return numerical values, the second set is just procedures which do something and return nothing .

The program starts by calling the main( ) function which controls the flow of the program. As shown in the flow chart the main( ) first calls the function data( ) which reads the number of finite elements (NELE), the tolerance and the order of the series of the flexural rigidity and mass per unit length (SORD), The data( ) function calls anther function named allvar( ) which allocates the stiffness matrix [K], the mass matrix [M], the matrix which will contain the inverse of the stiffness matrix [mat], the vectors of the flexural rigidity and mass pear unit length series (vectors [A] and [B]), the vectors W and W1 which will contain the eigenvectors during the iteration in the function solve( ) and the vector w which will be used to save the eigenvalues (the natural frequencies). All of these matrices and vectors are allocated with the size required during the run of the program. The required size depend on the number of elements and the order of the



Fig(A.1) Flow chart for the computer program which solves the problem of free vibration of rotating beams with non-uniform cross-section using the finite element method

polynomials represent the cross section variation. If there isn't enough memory space the computer will print the massage "insufficient memory space " and terminates the program. After allocating the variables the data() function reads the values of the non-dimensional rotational speed(η), the root offset ratio (α), the tip mass ratio(μ) and the setting angle(θ) in addition to the coefficient of the non-dimensional flexural rigidity and the non-dimensional mass per unit length series.

The main( ) function calls the function init( ). This function initializes the global stiffness matrix [K] and mass matrix [M] to zero. Also the matrix [mat] is initialized to the identity matrix .

After that, the computer enters a loop in which it first calls the function elequa(n). This function calculates the element matrices [EK] and [EM]. In this function the function integration( ) is called. The first three enters of the function integration( ) are pointers to functions and the forth is the element number n. The function integration( ) finds the integration of the product of the three entered functions over the element length which its number is the fourth entry using Simpson's one-third rule. The entered functions are combinations of the shape functions and their first and second derivatives in addition to the non-dimensional flexural rigidity, non-dimensional mass per unit length functions and the function FX(x) which gives the moment of inertia of the part of the beam beyond the point x. All of these functions which are called by the function integration( ) are defined to be of double precision type. This is done to improve the accuracy of the values found by the function integration( ).

The main( ) calls the function filmat(n) which uses the element matrices that was found by the function elequa(n) to fill the global mass and stiffness matrices ([K] and [M]) according to the relations :-

$$M[2(n-1)+i][2(n-1)+j] += EM[i][j]$$
$$K[2(n-1)+i][2(n-1)+j] += EK[i][j]$$

After reaching the last element the computer gets out of the loop and calls the function bdcon( ). This function asked the user to enter the boundary condition number (BDCN). This number represents the number of rows and columns to be removed from the global matrices in addition to specify the boundary condition. This number is defined such that :-

[0] Flexible attachment

[1] Hinged root attachment

[2] Fixed root attachment

If the boundary condition number is zero the computer will ask the user to enter the translation and rotational stiffness ratios (β and γ). The translation and rotational stiffness ratios (β and γ) with the tip mass ratio (μ) and the boundary condition number (BDCN) are all used in the application of the boundary conditions on the systems equation.

Then the function invert( ) will be called. This function inverts the system stiffness matrix [K] . The $[K]^{-1}[M]$ is found and saved in the matrix [mat]. At this stage the matrix [mat] is ready to apply the matrix deflation method described to get the eigenvalues and eigenvectors .

The user is asked to enter the number of eigenvalues (NEV) required. The maximum value is related to the number of finite elements (NELE) and the boundary condition numbers (BCN) through the relation:-

$$(NEV)_{max} = 2(NELE+1) - BCN$$

If the user entered a value greater than $(NEV)_{max}$ the computer will not accept the entered value and the number of eigenvalues required is read (NEV) again.

The main() starts a new loop to get the eigenvalues and eigenvectors. The function solve(n) is first called. This function estimates the n'th eigenvalue and eigenvector using the iteration technique described previously. Next the main( ) calls the function normalize(n) which normalizes the eigenvector. After normalization, the normalized eigenvector is used to deflate the matrix [mat] through calling the function deflate(n).This loop will be terminated when the last required eigenvalue and eigenvector are found.

The function display( ) is then called. This function calculates the non-dimensional natural frequency from the relation :-

$$\frac{\omega}{\sqrt{(\rho A_0 L^4)/(EI_0)}} = \sqrt{\lambda - \eta^2 \sin^2(\theta)}$$

The data displayed are the eigenvalues and the eigenvectors. The eigenvectors are normalized so that the maximum value of the mode shape is unity. These results are displayed in tabulated form.Each table contains five eigenvalues with their corresponding eigenvectors as shown in the sample run Appendix (A) section (A.3).

Finally the main ( ) asks the user if he wants to save the results to an output file If yes, the function save( ) is called. This function sake the user to enter the output file name. An output file of that name is generated and the results are saved to it in a form similar to that used in the function display( ).

The function freme( ) is called to free the allocated memory space that was allocated by the function allvar( ).and the program reaches its end.

The Computer programe

```
# include "stdio.h"
# include "ctype.h"
# include "math.h"
# include "stdlib.h"

double N1();
double N2();
double N3();
double N4();
double D1N1();
double D1N2();
double D1N3();
double D1N4();
double D2N1();
double D2N2();
double D2N3();
double D2N4();
double EI();
double m();
double FX();
float integration();

void main();
void data();
void allva();
void init();
void elequa();
void filmat();
void bdcon();
void invert();
void solve();
void normalize();
void deflate();
void disply();
```

```
void save();
void Mdsh();
void frme();

int NELE , SORD , BCN , NEV ;
float alpha , beta , theta , mew , eta , gama , h , EPS ;
float **K , **M , **mat, *W , *W1 , *A , *B , *w ;
float EM[4][4] , EK[4][4] ;

#define BI 3.1415927
```

void main()

```
    {
      int n , i , j ;
      char ch ;
      data();
      init();
      for(n=1;n<=NELE;n++){
                          elequa(n);
                          filmat(n);
                          }
      bdcon();
      invert(BCN);
      printf("The Number of Natural Frequencies Required (NEV<=%d)=",
      2*(NELE+1)-BCN);
      do{
        scanf("%d",&NEV);
     }while(NEV>2*(NELE+1)-BCN);
      for(n=0;n<NEV;n++){
                          solve(n,BCN);
                          normalize(BCN);
                          deflate(n,BCN);
                         }
      disply();
      printf(" Save The Results to an Output File (Y/N)?");
      do{
        ch=getchar();
        }while(!((toupper(ch)=='Y')||(toupper(ch)=='N')));
      if(toupper(ch)=='Y')save();
      printf(" Esstemate  Mode Shapes (Y/N)?");
      do{
        ch=getchar():
        }while(!((toupper(ch)=='Y')||(toupper(ch)=='N')));
      if(toupper(ch)=='Y')Mdsh();
      frme();
    }
```

double N1(x)

```
      double x;
```

```
    {
        return(1-3*x*x/(h*h)+2*x*x*x/(h*h*h));
    }
```

double N2(x)

```
        double x;
    {
        return(x*(1-x/h)*(1-x/h));
    }
```

double N3(x)

```
        double x;
    {
        return(3*x*x/(h*h)-2*x*x*x/(h*h*h));
    }
```

double N4(x)

```
        double x;
    {
        return(x*(x*x/(h*h)-x/h));
    }
```

double D1N1(x)

```
        double x;
    {
        return(6*(x*x/(h*h)-x/h)/h);
    }
```

double D1N2(x)

```
        double x;
    {
        return(1-4*x/h+3*x*x/(h*h));
    }
```

double D1N3(x)

```
        double x;
    {
        return(-6*(x*x/(h*h)-x/h)/h);
    }
```

double D1N4(x)

```
        double x;
    {
        return(3*x*x/(h*h)-2*x/h);
    }
```

double D2N1(x)

```
    double x;
{
    return(12*(x/(h*h*h))-6/(h*h));
}
```

double D2N2(x)

```
    double x;
{
    return((-4+6*x/h)/h);
}
```

double D2N3(x)

```
    double x;
{
    return(-6*(2*x/h-1)/(h*h));
}
```

double D2N4(x)

```
    double x;
{
    return((-2+6*x/h)/h);
}
```

double EI(x)

```
    double x ;
{
    int i;
    double sum=0 , p=1 ;
    for(i=0;i<=SORD;i++){
                        sum+=A[i]*p;
                        p=p*x;
                        }
    return(sum);
}
```

double m(x)

```
    double x;
{
    int i;
    double sm=0 , p=1 ;
    for(i=0;i<=SORD;i++){
```

```
                        sm+=B[i]*p;
                        p=p*x;
                        }
        return(sm);
    }
```

float integration(f1,f2,f3,ele)

```
        double (*f1)(), (*f2)() , (*f3)() ;
        int ele;
    {
        double dx , sum=0.0 , m ;
        int i;
        dx=h/20;
        for(i=1;i<20;i++){
                if(i%2)m=4.0;
                else m=2.0 ;
                sum+=m*(*f1)((ele-1)*h+i*dx)*(*f2)(i*dx)*(*f3)(i*dx);
                        }
        sum+= (*f1)((ele-1)*h)*(*f2)(0.0)*(*f3)(0.0)+(*f1)(ele*h)*
            (*f2)(h)*(*f3)(h) ;
        return((float)(dx*sum/3.0));
    }
```

double FX(x)

```
        float x;
    {
        int i;
        double sum=0.0 ,p=x ;
        for(i=0;i<=SORD;i++){
                        sum+=(B[i]*alpha/(i+1))*(1-p);
                        p*=x;
                        sum+=(B[i]/(i+2))*(1-p);
                        }
        return(sum+mew*(alpha+1));
    }
```

void data()

```
        {
        float f;
        int i;
        puts(" Enter The Folowing Data :-");
        printf("No. of Elements=");
        scanf("%d",&NELE);
        printf("The Tolerance (EPS)=");
        scanf("%f",&EPS);
        h=1.0/NELE;
        printf("The Order of THE m(x) and EI(x) Series=");
```

```
scanf("%d",&SORD);
allva();
printf("Alpha=");
scanf("%f",&alpha);
printf("Eta=");
scanf("%f",&eta);
printf("Mew=");
scanf("%f",&mew);
printf("Setting Angle (Deg.)=");
scanf("%f",&theta);
puts("Enter The coefficients of the f(x) Series:");
for(i=0;i<=SORD;i++){
        printf("A%d=",i);
        scanf("%f",&f);
        A[i]=f;
            }
puts("Enter The coefficients of the g(x) Series:");
for(i=0;i<=SORD;i++){
        printf("B%d=",i);
        scanf("%f",&f);
        B[i]=f;
            }
}
```

void allva()

```
{
 int i ;
        K=(float **)malloc(2*(NELE+1)*sizeof(float *));
        M=(float **)malloc(2*(NELE+1)*sizeof(float *));
        mat=(float **)malloc(2*(NELE+1)*sizeof(float *));
        if((M==0)||(K==0)||(mat==0)){
                printf("insuficient memory space");
                exit(0);
                        }
 for(i=0;i<=2*NELE+1;i++){
        K[i]=(float *)malloc(2*(NELE+1)*sizeof(float));
        M[i]=(float *)malloc(2*(NELE+1)*sizeof(float));
        mat[i]=(float *)malloc(2*(NELE+1)*sizeof(float));
        if((M[i]==0)||(K[i]==0)||(mat[i]==0)){
                printf("insuficient memory space");
                exit(0);
                        }
            }
 A=(float *)malloc((SORD+1)*sizeof(float));
 B=(float *)malloc((SORD+1)*sizeof(float));
 W=(float *)malloc(2*(NELE+1)*sizeof(float));
 W1=(float *)malloc(2*(NELE+1)*sizeof(float));
 w=(float *)malloc(2*(NELE+1)*sizeof(float));
 if((A==0)||(B==0)||(W==0)||(W1==0)||(w==0)){
                printf("insuficient memory space");
```

```
                    exit(0);
                          }
    }

void init()

    {
       int , j ;
       for(i=0;i<=2*NELE+1;i++){
                 for(j=0;j<=2*NELE+1;j++){
                        K[i][j]=0.0;
                        M[i][j]=0.0;
                        mat[i][j]=0.0;
                              }
                        mat[i][i]=1.0;
                        }
    }

void elequa(n)
       int n ;
    {

       EK[0][0]=integration(EI,D2N1,D2N1,n)+
              eta*eta*integration(FX,D1N1,D1N1,n);
       EK[0][1]=EK[1][0]=integration(EI,D2N1,D2N2,n)+
                     eta*eta*integration(FX,D1N1,D1N2,n);
       EK[0][2]=EK[2][0]=integration(EI,D2N1,D2N3,n)+
                eta*eta*integration(FX,D1N1,D1N3,n);
       EK[0][3]=EK[3][0]=integration(EI,D2N1,D2N4,n)+
                     eta*eta*integration(FX,D1N1,D1N4,n);
       EK[1][1]=integration(EI,D2N2,D2N2,n)+
              eta*eta*integration(FX,D1N2,D1N2,n);
       EK[1][2]=EK[2][1]=integration(EI,D2N2,D2N3,n)+
              eta*eta*integration(FX,D1N2,D1N3,n);
       EK[1][3]=EK[3][1]=integration(EI,D2N2,D2N4,n)+
              eta*eta*integration(FX,D1N2,D1N4,n);
       EK[2][2]=integration(EI,D2N3,D2N3,n)+
              eta*eta*integration(FX,D1N3,D1N3,n);
       EK[2][3]=EK[3][2]=integration(EI,D2N3,D2N4,n)+
              eta*eta*integration(FX,D1N3,D1N4,n);
       EK[3][3]=integration(EI,D2N4,D2N4,n)+
              eta*eta*integration(FX,D1N4,D1N4,n);

       EM[0][0]=integration(m,N1,N1,n);
       EM[0][1]=EM[1][0]=integration(m,N1,N2,n);
       EM[0][2]=EM[2][0]=integration(m,N1,N3,n);
       EM[0][3]=EM[3][0]=integration(m,N1,N4,n);
       EM[1][1]=integration(m,N2,N2,n);
       EM[1][2]=EM[2][1]=integration(m,N2,N3,n);
       EM[1][3]=EM[3][1]=integration(m,N2,N4,n);
       EM[2][2]=integration(m,N3,N3,n);
       EM[2][3]=EM[3][2]=integration(m,N3,N4,n);
```

```
        EM[3][3]=integration(m,N4,N4,n);
    }

void filmat(n)

        int n;
    {
        int i , j ;
        for(i=0;i<=3;i++){
            for(j=0;j<=3;j++){
                        K[2*(n-1)+i][2*(n-1)+j]+=EK[i][j];
                                M[2*(n-1)+i][2*(n-1)+j]+=EM[i][j];
                            }
                        }
    }

void bdcon()

    {
        puts("   The Boundary Conditons at The Root\n");
        printf("\n\n\n\n");
        printf("        [0] Flexible Root Attachment\n");
        printf("        [1] Hinged Root Condition\n");
        printf("        [2] Fixed Root Condition\n\n\n");
        printf("  The Boundary Condition Number =");
        do{
          scanf("%d",&BCN);
         }while((BCN>3)||(BCN<0.0));
        if(!BCN){
                printf("The Rotational and Translational Stiffness:\n");
                printf("Beta=");
                scanf("%f",&beta);
                printf("Gama=");
                scanf("%f",&gama);
                K[0][0]+=beta;
                K[1][1]+=gama;
                }
         M[2*NELE][2*NELE]+=mew;
         M[2*NELE+1][2*NELE+1]+=mew;
    }

void invert(n)

        int n ;
    {
        float  A ;
        int i , j , k ;

        for(i=n;i<=2*NELE;i++){
                    for(j=i+1;j<=2*NELE+1;j++){
                            A=K[j][i]/K[i][i];
                            for(k=n;k<=2*NELE+1;k++){
```

```
                                        K[j][k]-=A*K[i][k];
                                        mat[j][k]-=A*mat[i][k];
                                                }
                                          }
                              }
      for(i=2*NELE+1;i>n;i--){
                 for(j=i-1;j>=n;j--){
                           A=K[j][i]/K[i][i];
                       for(k=n;k<=2*NELE+1;k++){
                                               K[j][k]-=A*K[i][k];
                                               mat[j][k]-=A*mat[i][k];
                                              }
                                  }
                              }
      for(i=n;i<=2*NELE+1;i++){
                                  A=K[i][i];
       for(j=n;j<=2*NELE+1;j++){
                                     K[i][j]=K[i][j]/A;
                                     mat[i][j]=mat[i][j]/A;
                                    }
                                 }
      for(i=n;i<=2*NELE+1;i++){
               for(j=n;j<=2*NELE+1;j++){
                                         A=0.0;
                         for(k=n;k<=2*NELE+1;k++)A+=mat[i][k]*M[k][j];
                             K[i][j]=A;
                                       }
                                 }
      for(i=n;i<=2*NELE+1;i++){
                   for(j=n;j<=2*NELE+1;j++)mat[i][j]=K[i][j];
                                 }
  }
```

void solve(n,m)

```
      int n , m ;
  {
      int i , j ;
      float A , A1=1.0 , s ;
      for(i=m;i<=2*NELE+1;i++)W1[i]=1.0;
      do{
        A=A1;
        for(i=m;i<=2*NELE+1;i++){
                  s=0.0;
                  for(j=m;j<=2*NELE+1;j++)s+=mat[i][j]*W1[j];
                  W[i]=s;
                                  }
        A1= fabs(W[m]);
        for(i=m;i<=2*NELE+1;i++)W[i]=W[i]/A1;
        for(i=m;i<=2*NELE+1;i++)W1[i]=W[i];
       }while(fabs(A-A1)>EPS);
      for(i=m;i<=2*NELE+1;i++){
```

```
                                        K[i][n]=W[i];
                                                }
        w[n]=A1;
    }
void normalize(n)
        int n ;
    {
        int i , j ;
        float s=0.0 , A=0.0 ;
        for(i=n;i<=2*NELE+1;i++){
            A=0.0;
            for(j=n;j<=2*NELE+1;j++)A+=W[j]*M[j][i];
            A*=W[i];
            s+=A;
                                        }
        A=(float) sqrt(1.0/s);
        for(i=n;i<=2*NELE+1;i++)W[i]=W[i]*A;
    }

void deflate(n,m)
        int n , m ;
    {
        int i , j , k ;
        float s ;
        for(i=m;i<=2*NELE+1;i++){
                    for(j=m;j<=2*NELE+1;j++){
                            s=0.0;
                            for(k=m;k<=2*NELE+1;k++)s+=W[i]*W[k]*M[k][j];
                            mat[i][j]-=w[n]*s;
                                            }
                                        }
    }
```

void disply()

```
    {
        int i,j,k,m,e;
        float a;
        for(i=0;i<NEV;i++){
            for(j=0;j<BCN;j++)K[j][i]=0.0;
                            }
        for(i=0;i<NEV;i++){
                a=K[2*NELE][i];
                for(j=0;j<=2*NELE+1;j++)K[j][i]=K[j][i]/a;
                            }
        a=eta*sin(BI*theta/180);
        printf(" Natural Frequenceis of a Rotating Beam With Non Uniform\n");
        printf("    Cross Section by the Finite Element Method\n\n\n");
        printf("____________________________________________________________\n\n");
        printf("for:\n");
        printf("Number of Elements (NELE)          = %d\n",NELE);
        printf("The Tolerance      (EPS)        = %f\n",EPS);
```

```
printf("     Eta                          = %f\n",eta);
printf("     Root Offset (Alpha)          = %f \n",alpha);
printf("     Tip Mss (Mew)                = %f \n",mew);
printf("     Setting Angle (Theta) in deg.= %f\n",theta);
if(BCN==1)printf("     With Hinged Root Condition .\n");
if(BCN==2)printf("     With Fixed Root Condition .\n");
if(BCN==0){
          printf("    With Flexible Attachment with:\n");
          printf("         Beta = %f\n",beta);
          printf("         Gama = %f\n",gama);
         }
printf("________________________________________________________\n\n");
printf("  The Coefficients of the g(x) and Flexural Rigidity f(x)\n");
printf(" series with A(i) and B(i) are :-\n");
printf("  i   A(i)     B(i) \n");
printf("________________________\n");
for(i=0;i<=SORD;i++)printf("  %d   %5.2f   %5.2f\n",i,A[i],B[i]);
printf("________________________________________________________\n\n");
m=NEV/5;
if(NEV%5)m+=1;
printf("Press Enter to Continue \n");
getchar();
getchar();
for(i=1;i<=m;i++){
if(i==m)e=NEV;
else e=5*i;
printf("\n\n\n   Eigen Values and Eigen Vectors\n");
printf("             Table(%d)\n",i);
printf("________________________________________________________\n");
printf("w    |");
for(j=5*(i-1);j<=e-1;j++)printf("  w%d=     ",j+1);
printf("\n________________________________________________________\n");
printf("   i |");
for(j=5*(i-1);j<e;j++)printf("%8.5f  ",sqrt(1.0/w[j]-a*a));
printf("\n________________________________________________________\n");
for(k=0;k<2*(NELE+1);k++){
printf(" %2d  |",k+1);
for(j=5*(i-1);j<=e-1;j++)printf(" %8.4f ",K[k][j]);
printf("\n");
if(!((k+1)%10)){
          printf("Press Enter to Continue \n");
          getchar();
          }
                         }
printf("________________________________________________________\n");


                    }
}
```

void save()

```
{
int i,j,k,m,e;
float a;
char Name[11];
FILE * fp;
printf("Enter The Output File Name (10 Chara.):\n");
getchar();
gets(Name);
if((fp=fopen(Name,"w"))==NULL){
puts("Cannot Open Output File\n");
exit(0);
                              }
a=eta*sin(BI*theta/180);
fprintf(fp," Natural Frequenceis of a Rotating Beam With Non Uniform\n");
fprintf(fp,"    Cross Section by the Finite Element Method\n\n\n");
fprintf(fp,"____________________________________________________________\n\n");
fprintf(fp,"for:\n");
fprintf(fp,"Number of Elements (NELE)        = %d\n",NELE);
fprintf(fp,"The Tolerance      (EPS)        = %f\n",EPS);
fprintf(fp,"     Eta                        = %f\n",eta);
fprintf(fp,"     Root Offset (Alpha)        = %f \n",alpha);
fprintf(fp,"     Tip Mss (Mew)              = %f \n",mew);
fprintf(fp,"     Setting Angle (Theta) in deg.= %f\n",theta);
if(BCN==1)fprintf(fp,"     With Hinged Root Condition .\n");
if(BCN==2)fprintf(fp,"     With Fixed Root Condition .\n");
if(BCN==0){
fprintf(fp,"     With Flexible Attachment with:\n");
fprintf(fp,"          Beta = %f\n",beta);
fprintf(fp,"          Gama = %f\n",gama);
          }
fprintf(fp,"____________________________________________________________\n\n");
fprintf(fp,"  The Coefficients of the g(x) and Flexural Rigidity f(x)\n");
fprintf(fp," series with A(i) and B(i) are :-\n");
fprintf(fp," i   A(i)    B(i) \n");
fprintf(fp,"______________________\n");
for(i=0;i<=SORD;i++)fprintf(fp," %d   %5.2f   %5.2f\n",i,A[i],B[i]);
fprintf(fp,"____________________________________________________________\n\n");
m=NEV/5;
if(NEV%5)m+=1;
for(i=1;i<=m;i++){
if(i==m)e=NEV;
else e=5*i;
fprintf(fp,"\n\n\n   Eigen Values and Eigen Vectors\n");
fprintf(fp,"                Table(%d)\n",i);
fprintf(fp,"____________________________________________________________\n");
fprintf(fp,"   w |");
for(j=5*(i-1);j<=e-1;j++)fprintf(fp,"  w%d=     ",j+1);
fprintf(fp,"\n____________________________________________________________\n");
fprintf(fp,"   i |");
for(j=5*(i-1);j<=e-1;j++)fprintf(fp,"%8.5f ",sqrt(1.0/w[j]-a*a));
fprintf(fp,"\n____________________________________________________________\n");
for(k=0;k<2*(NELE+1);k++){
fprintf(fp," %2d  |",k+1);
```

```
for(j=5*(i-1);j<=e-1;j++)fprintf(fp," %8.4f ",K[k][j]);
fprintf(fp,"\n");
                        }
fprintf(fp,"____________________________________________________\n");
                        }
 fclose(fp);
}
```

void Mdsh()

```
{
  int i , j , NMS ;
  float dx , Fx , x ;
  char Name[11];
  FILE * fp1;
  printf("Enter The Output File Name (10 Chara./.MSH/)\n");
  getchar();
  gets(Name);
  if((fp1=fopen(Name,"w"))==NULL){
  puts("Cannot Open Output File\n");
  exit(0);
                                  }
  printf("Enter The Number of Mode Shapes Required(NMS<=%d)=",NEV);
  scanf("%d",&NMS);
  printf("Enter the increment dx:");
  scanf("%f",&dx);
  x=0.0;
  while(x<=1.0){
  printf("%4.2f",x);
  fprintf(fp1,"%4.2f",x);
  for(j=0;j<NMS;j++){
  i=0;
  con: if((x>=(h*i))&&(x<=(h*(i+1)))){
          Fx=N1(x-i*h)*K[2*i][j]+N2(x-i*h)*K[2*i+1][j]+
          N3(x-i*h)*K[2*(i+1)][j]+N4(x-i*h)*K[2*(i+1)+1][j];
          printf(" %6.4f",Fx);
          fprintf(fp1," %6.4f",Fx);
                                  }
      else{
         i+=1;
         goto con;
         }
                  }
 printf("\n");
 fprintf(fp1,"\n");
 x+=dx;
   };
 fclose(fp1);

}
```

```
void frme()
    {
        int i , j ;
        for(i=0;i<=2*NELE+1;i++){
                                    free(mat[i]);
                                    free(K[i]);
                                    free(M[i]);
                                   }
        free(mat);
        free(K);
        free(M);
        free(A);
        free(B);
        free(w);
        free(W1);
        free(W);
    }
```

A.3 Sample Run

Natural Frequenceis of a Rotating Beam With Non Uniform Cross Section by the Finite Element Method

---

```
for:
Number of Elements (NELE)        = 15
The Tolerance      (EPS)          = 0.000001
    Eta                            = 1.000000
    Root Offset (Alpha)           = 0.000000
    Tip Mss (Mew)                  = 0.000000
    Setting Angle (Theta) in deg. = 0.000000
    With Fixed Root Condition .
```

The Coefficients of the g(x) and Flexural Rigidity f(x) series with A(i) and B(i) are :-

i	A(i)	B(i)
0	1.000000e+00f	1.000000e+00
1	-1.000000e+00f	-1.000000e+00
2	5.000000e-01f	5.000000e-01
3	-1.666670e-01f	-1.666670e-01
4	4.166667e-02f	4.166667e-02
5	-8.333340e-03f	-8.333340e-03
6	1.388889e-03f	1.388889e-03
7	-1.941270e-04f	-1.941270e-04
8	2.481590e-05f	2.481590e-05
9	-2.755730e-06f	-2.755730e-06
10	2.755730e-07f	2.755730e-07
11	-2.505211e-08f	-2.505211e-08
12	2.087680e-09f	2.087680e-09
13	-1.605900e-10f	-1.605900e-10
14	1.147070e-11f	1.147070e-11
15	-7.647200e-13f	-7.647200e-13
16	4.779477e-14f	4.779477e-14
17	-2.811457e-15f	-2.811457e-15
18	1.561921e-16f	1.561921e-16

Eigen Values and Eigen Vectors
Table(1)

w	w1=	w2=	w3=	w4=	w5=
i	4.86263	24.31463	64.01997	123.55725	205.03012
1	0.00000	-0.00000	0.00000	-0.00000	0.00000
2	0.00000	-0.00000	0.00000	-0.00000	0.00000
3	0.00638	-0.02999	0.07361	-0.13168	0.20108
4	0.19012	-0.85859	2.00744	-3.37729	4.76498
5	0.02517	-0.10861	0.23971	-0.37348	0.47561
6	0.37201	-1.45331	2.76211	-3.33053	2.47054
7	0.05576	-0.21730	0.41432	-0.51486	0.44905

8	0.54400	-1.75729	2.28389	-0.58033	-3.40826
9	0.09744	-0.33617	0.52217	-0.42965	0.06851
10	0.70436	-1.75828	0.82088	3.11170	-7.23071
11	0.14938	-0.44517	0.51226	-0.12752	-0.38213
12	0.85144	-1.46447	-1.15582	5.58762	-5.25696
13	0.21064	-0.52556	0.37013	0.25316	-0.52764
14	0.98369	-0.90763	-3.04064	5.31635	1.28396
15	0.28018	-0.56154	0.12224	0.51703	-0.22767
16	1.09981	-0.14268	-4.23842	2.20639	7.08661
17	0.35689	-0.54160	-0.17002	0.51622	0.28579
18	1.19877	0.75583	-4.32247	-2.27540	7.14728
19	0.43960	-0.45971	-0.42608	0.23517	0.58498
20	1.28006	1.70039	-3.15819	-5.80154	1.08543
21	0.52715	-0.31586	-0.56773	-0.19069	0.40091
22	1.34359	2.60019	-0.95160	-6.37988	-6.26141
23	0.61836	-0.11587	-0.54076	-0.53575	-0.13627
24	1.38995	3.37226	1.79905	-3.43363	-8.64087
25	0.71212	0.12952	-0.33021	-0.59603	-0.58047
26	1.42055	3.95404	4.44581	1.83397	-3.55712
27	0.80749	0.40642	0.03638	-0.29238	-0.51868
28	1.43765	4.31607	6.39988	7.06056	5.55536
29	0.90366	0.70046	0.50069	0.29575	0.10531
30	1.44440	4.47398	7.36264	10.11994	12.32299
31	1.00000	1.00000	1.00000	1.00000	1.00000
32	1.44543	4.49988	7.53394	10.72158	13.81896

Appendix B

The Power Series Method

B.1 Power Series Method For Solving Differential Equations

The power series method is the only method which can be used to solve differential equations with variable coefficients exactly. reference [16], [19] and [22] explained the series method for solving second order differential equations with variable coefficients , but this method could be extended to include linear differential equations of any order.

To explain this solution procedure briefly around some point x_0 for an n'th order differential equation given by :-

$$f_n(x)\frac{d^n y}{dx^n}+f_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\ldots+f_0(x)y=0$$

where $f_i(x)$ i=0,1,2,3,...,n are functions of the independent variable x , they are all continuos and have a polynomial form, for functions which are not polynomial it is assumed that they could be approximated by a polynomial by using Taylor series. The point $x=x_0$ may be classified as :-

1- Ordinary point if $f_n(x_0) \neq 0$, and since $f_n(x)$ is assumed to be continuos then there is an interval around the point x_0 where a unique olution is exist .

2- If $f_n(x_0)=0$ the point is called singular and we will have one of the two cases :-

a-if all $f_i(x_0)=0$ for $i=0,1,2,3,...,n-1$, then the point x- is called a regular singular point and we can find one solution at least .

b- if at least one of $f_i(x_0) \neq 0$ then the point $x=x_0$ is called singular point and no solution could be found around it

The differential equation shown above can be written in the form:-

$$\frac{d^n y}{dx^n} + q_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + ... + q_0(x)y = 0$$

.where :-

$$q_i = \frac{f_i(x)}{f_n(x)}$$

Ref[17] shows that if q_i is analytic i.e smooth and continuose, for the range x_0 to x_1 then for any value of x in this range thire exist an n solution of a power series form which at least convergant in this interval.

The form of the solution which will be assumed depends on the case considered. For the ordinary case we assume that the solution takes the form :-

$$y(x) = \sum_{n=0}^{\infty} C_n (x - x_0)^n$$

and for the regular singular case at least one solution can be found which is of the form :-

$$y(x) = \sum_{n=0}^{\infty} C_n (x - x_0)^{n+r}$$

where r is a constant which will be found from the differential equation after substituting the expected form of solution so that the differential equation is satisfied .For singuler points no solution can be found at all.

The next step after substituting the expected form of solution is to find the coefficients C_n and this could be done by first finding the recurrence formula which takes the form :-

$$C_n = F(C_{n-1}, C_{n-2}, \ldots, C_0)$$

The complexity of the recurrence formula depends on the order of the differential equation and the order of the polynomial functions $f_i(x)$. After getting the recurrence formula the next step is to find the coefficients C_n in terms of the arbitrary coefficients which their number is equal to the order of the differential equation .

Finally the boundary conditions of the differential equation are used to get the required solution

In spite of its being simple in concept and application the series method is always avoided in solving differential equations especially in the case of eigen-value problems . this due to the fact that in most of the cases we face, the recurrence formula we get is always complex which is as stated above depends on the order of the differential equation and the form of the coefficient functions , so that it is very difficult to solve for the coefficients explicitly, but if we find a way to avoid this difficulty then the series method will be one of the most powerful method for solving linear differential equations.

B.3 Solution of the Problem of Free Vibration of Rotating Beams with Non-Uniform Cross-Section by the Series Method (Computer Program)

The computer program shown in bellow solves the problem of free vibration of rotating beams with non-uniform cross section with its most general case given by the system equation derived in chapter two with its boundary conditions. As stated chapter four the idea used in this program is to find the value of the non-dimensional natural frequency which makes the determinate of the boundary condition matrix zero for the case under consideration.

Fig(B.1) shows a flow chart for the computer program shown bellow. This program was written in C Language. In this program two data structures are defined. The first is the series coefficients type (SCO) which contains four double precision variables (C_1, C_2, C_1 and C_3). It was given the name SERIES..A vector of size equal to the approximating function order (FORD) is to be allocated during the run of this program. This vector will be used to save the coefficients of the approximating function W..

The second defined data structure is the (INTTYPE). It has two float variables. A vector of size equal the number of natural frequencies required will also be allocated during the run of the program. This vector will be used to save the intervals in which the function search( ) will detect a change in the sign of the determent of the boundary condition matrix. In other words it will contain the intervals in which there are non-dimensional natural frequencies.

As shown in figure (B.1) the program starts by calling the function main() which controls the flow of the program. The main( ) function first calls the data( ) function which reads the approximating function order (FORD), the tolerance (EPS), the order of the cross-section properties variation series (SORD), the root offset ratio (α), the nun-dimensional rotational speed (η), the tip mass ratio (μ) and the number of natural frequencies required (NEV). The data( ) function calls anther function called allva( ).This function allocates the required variables in the required size. The coefficients of the cross-section variation properties series($f(\xi)$ and $g(\xi)$)are read.

The main( ) function then calls the function bdcon(). This function reads the boundary condition number (BCN) which was defined in the same way as that used in the finite element method program. If the boundary condition number (BCN) is zero the user will be asked to enter the non-dimensional translation and rotational stiffness.

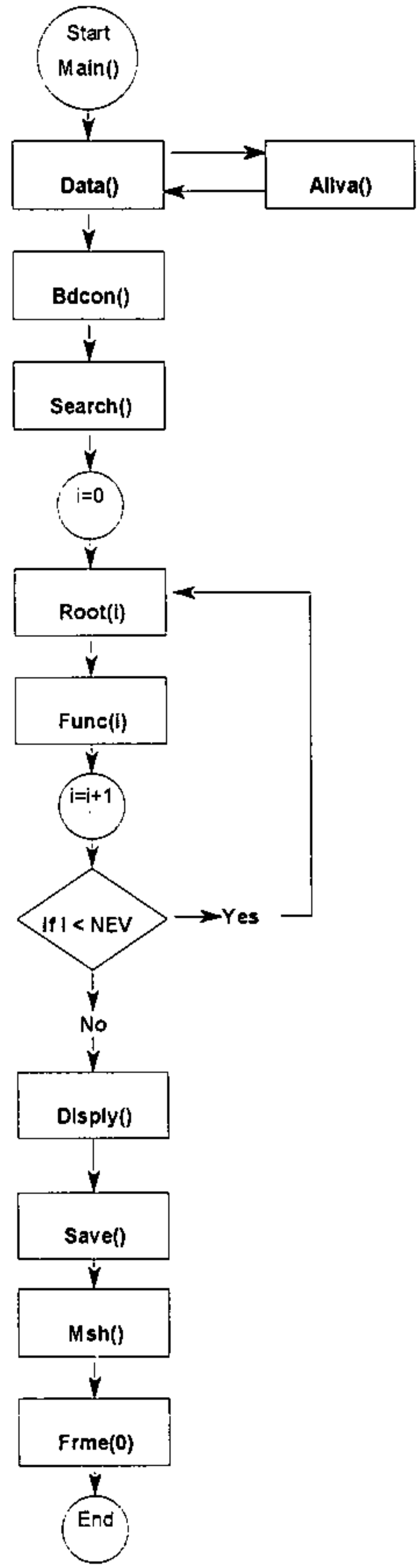
The function search( ) is then called. This function makes search on the non-dimensional natural frequency to find the intervals in which there is a root. The search procedure based on detecting a sign change or zero determent for the boundary condition matrix [a]. The interval for which the search procedure is performed on starts from zero and continue until getting the interval of the last non-dimensional natural frequency required.

For each assumed value of the natural frequency the search( ) function calls the initi( ) function which initialize the coefficient vector to zero, the coef( ) function which estimates the coefficients of the W(x) function. Then the appbdcon( ) function is called. This function apply the boundary conditions and generates the boundary conditions matrix. Finally the det( )

function is called. This function use the Guasian alimentation method to get the determent of the boundary condition matrix.

Using the value of the determent found from the det( ) function if the determent is zero or it change sign from the determinate of the non-dimensional natural frequency assumed before the resent one, the interval is saved in the RTINTE vector. Other wise the resent assumed natural frequency and the associated determent are saved and anew natural frequency is assumed and the previously described procedure is applied again. This process is repeated until the interval of the last required natural frequency is found.

After this, The computer enters a loop to get the exact values of the required non-dimensional natural frequencies and mode shapes which were detected by the search( ) function. In this loop first the main() calls the function root() which uses the bisection method to get the i'th natural frequency from the i'th interval detected by the search method, then the



Fig(B.1) Flow chart for the computer program which solves the problem of free vibration of rotating beams with non-uniform cross-section using the series method

Func() function finds the mode shape by first generates the boundary conditions matrix and assumes one of the arbitrary constants to be one and solves for the remaining constants in term of the one assumed to be one by using the Guasian elimination method. By using these values the mode shape function is reduced to one function with one arbitrary constant.

The function display( ) is called to show on the screen the natural frequencies and coefficients of the mode shape function in tabulated form. Each table contains five natural frequencies and their corresponding mode shapes.

If the user wants to save the results to an output file the save( ) function will be called. This function will ask the user to enter the output file name. The results are saved to an output file of that name in the same form used in the display function.

A function called Msh( ) which calculates the mode shapes at discreet points and saves them to an output file is then called. The mode shapes are saved in a form that can be used by a graphics program to draw the mode shapes. This done to save time in analysis needed in next work.

Finally, a function called frme( ) is called to free the memory allocated for the different variables needed during the execution of this program.

Computer Program

```
# include "stdio.h"
# include "ctype.h"
# include "math.h"
# include "stdlib.h"

float EI();
float D1EI();
float FX();
void coef();
void W();
void D1W();
void D2W();
void D3W();
float Fun();
float det();

void data();
void bdcon();
void allva();
void init();
void appbdcon();
void search();
void root();
void FUNC();
void disply();
void save();
void Mdsh();
void frme();


int SORD , FORD , BCN , NEV ;
float alpha , beta=0.0 , gama=0.0 , eta , mew , theta ;
float *A , *B , *w , *F , **K ;
float mat[4][4] , EPS ;
FILE * fp;
FILE * fd;

struct SCO {
          double C0;
          double C1;
          double C2;
          double C3;
          };
struct INTTYPE {
               float x0 ;
```

```
                float x1 ;
              };
typedef struct SCO SERIES;
typedef struct INTTYPE INT,
SERIES *c , *pointer1 , W1 , W2 , W3 , W4 ;
INT  *RTINTE;

#define BI 3.1415927
```

void main()

```
        {
        int i;
        char ch ;

        data();
        bdcon();
        search();
        for(i=0;i<NEV;i++){
                            root(i);
                            FUNC(i);
                            }
      disply();
        do{
          printf(" Save The Results to an Output File (Y/N)?");
          ch=getchar();
         }while(!((toupper(ch)=='Y')||(toupper(ch)=='N')));
        if(toupper(ch)=='Y')save();
        do{
          printf("Esstemate Mode Shapes and\n");
          printf("Save them to Output File (Y/N)?");
          ch=getchar();
         }while(!((toupper(ch)=='Y')||(toupper(ch)=='N')));
        if(toupper(ch)=='Y')Mdsh();
        frme();
        }
```

float EI(x)

```
        float x ;
     {
        int i;
        float sum=0 , p=1 ;
        for(i=0;i<=SORD;i++){
                            sum+=A[i]*p;
                            p=p*x;
                            }
        return((float)sum);
```

```
    }
```

float D1EI(x)

```
        float x ;
    {
        int i;
        float sum=0 , p=1 ;
        for(i=1;i<=SORD;i++){
                                sum+=i*A[i]*p;
                                p=p*x;
                               }
        return((float)sum);
    }
```

void W(x,pointer)

```
        float x;
        SERIES *pointer;
    {
        int i;
        float sum0=0,sum1=0,sum2=0,sum3=0,p=1;
        for(i=0;i<=FORD;i++){
                                sum0+=c[i].C0*p;
                                sum1+=c[i].C1*p;
                                sum2+=c[i].C2*p;
                                sum3+=c[i].C3*p;
                                p*=x;
                               }
        (*pointer).C0=sum0;
        (*pointer).C1=sum1;
        (*pointer).C2=sum2;
        (*pointer).C3=sum3;
    }
```

void D1W(x,pointer)

```
        float x;
        SERIES *pointer;
    {
        int i;
        float sum0=0,sum1=0,sum2=0,sum3=0,p=1;
        for(i=1;i<=FORD;i++){
                                sum0+=i*c[i].C0*p;
                                sum1+=i*c[i].C1*p;
                                sum2+=i*c[i].C2*p;
                                sum3+=i*c[i].C3*p;
```

```
                    p*=x;
                    }
    (*pointer).C0=sum0;
    (*pointer).C1=sum1;
    (*pointer).C2=sum2;
    (*pointer).C3=sum3;
}
```

void D2W(x,pointer)

```
    float x;
    SERIES *pointer;
{
    int i;
    float sum0=0,sum1=0,sum2=0,sum3=0,p=1;
    for(i=2;i<=FORD;i++){
                    sum0+=i*(i-1)*c[i].C0*p;
                    sum1+=i*(i-1)*c[i].C1*p;
                    sum2+=i*(i-1)*c[i].C2*p;
                    sum3+=i*(i-1)*c[i].C3*p;
                    p*=x;
                    }
    (*pointer).C0=sum0;
    (*pointer).C1=sum1;
    (*pointer).C2=sum2;
    (*pointer).C3=sum3;
}
```

void D3W(x,pointer)

```
    float x;
    SERIES *pointer;
{
    int i;
    float sum0=0,sum1=0,sum2=0,sum3=0,p=1;
    for(i=3;i<=FORD;i++){
                    sum0+=i*(i-1)*(i-2)*c[i].C0*p;
                    sum1+=i*(i-1)*(i-2)*c[i].C1*p;
                    sum2+=i*(i-1)*(i-2)*c[i].C2*p;
                    sum3+=i*(i-1)*(i-2)*c[i].C3*p;
                    p*=x;
                    }
    (*pointer).C0=sum0;
    (*pointer).C1=sum1;
    (*pointer).C2=sum2;
    (*pointer).C3=sum3;
}
```

```
float FX(x)
        float x;
    {
        int i;
        float sum=0.0 ,p ;
        p=x;
        for(i=0;i<=SORD;i++){
                                sum+=(B[i]*alpha/(i+1))*(1-p);
                                p*=x;
                                sum+=(B[i]/(i+2))*(1-p);
                                }
        return((float)(sum+mew*(alpha+1.0)));
    }


float Fun(x,n)
        int n ;
        float x ;
    {
        int i;
        float sum=0 , p=1 ;
        for(i=0;i<=FORD;i++){
                                sum+=K[n][i]*p;
                                p=p*x;
                                }
        return((float)sum);
    }


void data()
        {
         float f;
         int i;
         puts(" Enter The Folowing Data :-");
         printf("The Order of The W Function=");
         scanf("%d",&FORD);
         printf("The Tolerance (EPS)=");
         scanf("%f",&EPS);
         printf("The Order of THE m(x) and EI(x) Series=18\n");
         SORD=18;
         printf("Alpha=");
         scanf("%f",&alpha);
         printf("Eta=");
         scanf("%f",&eta);
         printf("Mew=");
         scanf("%f",&mew);
         printf("Setting Angle (Deg.)=");
```

```
scanf("%f",&theta);
printf("The No. of Eigenvalues Required=");
scanf("%d",&NEV);
allva();
 if((fd=fopen("DATA.DATA","r"))==NULL){
 puts("Cannot Open Input File\n");
 exit(0);
                         }

for(i=0;i<=SORD;i++){
        fscanf(fd,"%f",&f);
        A[i]=f;
            }
for(i=0;i<=SORD;i++){
        fscanf(fd,"%f",&f);
        B[i]=f;
            }

}
```

void allva()

```
{
  int i ;
  c=(SERIES *)malloc((FORD+1)*sizeof(SERIES));
  A=(float *)malloc((SORD+1)*sizeof(float));
  B=(float *)malloc((SORD+1)*sizeof(float));
  w=(float *)malloc(NEV*sizeof(float));
  RTINTE=(INT*)malloc(NEV*sizeof(INT));
  F=(float *)malloc((FORD+1)*sizeof(float));
  K=(float **)malloc((NEV+1)*sizeof(float*));
  if((A==0)||(B==0)||(c==0)||(w==0)||(RTINTE==0)||(F==0)||(K==0)){
                    printf("insuficient memory space\n");
                    exit(0);
                                }
for(i=0;i<=(NEV+1);i++){
                K[i]=(float *)malloc((FORD+1)*sizeof(float));
                if(K[i]==0){
                        printf("insuficient memory space\n");
                        exit(0);
                        }
                }

}
```

```
void init()
    {
      int i ;
      for(i=0;i<=FORD;i++){
                          c[i].C0=0.0;
                          c[i].C1=0.0;
                          c[i].C2=0.0;
                          c[i].C3=0.0;
                          }
      c[0].C0=1.0;
      c[1].C1=1.0;
      c[2].C2=1.0;
      c[3].C3=1.0;
    }


void coef(lamda)
      float lamda ;
    {
      int n , k , m ;
      double s;
      for(n=0;n<=FORD-4;n++){

      if(n+2>SORD)m=SORD;
      else m=n+2;

      for(k=1;k<=m;k++){
                         s=-A[k]*(n-k+4)*(n-k+3)*(n+2)*(n+1);
                         c[n+4].C0+=s*c[n-k+4].C0;
                         c[n+4].C1+=s*c[n-k+4].C1;
                         c[n+4].C2+=s*c[n-k+4].C2;
                         c[n+4].C3+=s*c[n-k+4].C3;
                         }

         s=0.0;
         for(k=0;k<=SORD;k++)s+=B[k]*(alpha/(k+1)+1.0/(k+2));
         s=eta*eta*(n+2)*(n+1)*(s+mew*(alpha+1.0));
         c[n+4].C0+=s*c[n+2].C0;
         c[n+4].C1+=s*c[n+2].C1;
         c[n+4].C2+=s*c[n+2].C2;
         c[n+4].C3+=s*c[n+2].C3;

         if(n<=SORD)m=n;
         else m=SORD;
         for(k=0;k<=m;k++){
                    s=-B[k]*eta*eta*alpha*(n-k+1)*(n+1)/(k+1);
                    c[n+4].C0+=s*c[n-k+1].C0;
                    c[n+4].C1+=s*c[n-k+1].C1;
```

```
                    c[n+4].C2+=s*c[n-k+1].C2;
                    c[n+4].C3+=s*c[n-k+1].C3;
                          }

            for(k=0;k<=m;k++){
                    s=-B[k]*(eta*eta*(n-k)*(n+1)/(k+2)-lamda);
                    c[n+4].C0+=s*c[n-k].C0;
                    c[n+4].C1+=s*c[n-k].C1;
                    c[n+4].C2+=s*c[n-k].C2;
                    c[n+4].C3+=s*c[n-k].C3;
                            }

          s=A[0]*(n+4)*(n+3)*(n+2)*(n+1);
          c[n+4].C0/=s;
          c[n+4].C1/=s;
          c[n+4].C2/=s;
          c[n+4].C3/=s;
        }
    }
```

void bdcon()

```
    {
      puts("   The Boundary Conditons at The Root\n");
      printf("\n\n\n\n");
      printf("        [0] Flexible Root Attachment\n");
      printf("        [1] Hinged Root Condition\n");
      printf("        [2] Fixed Root Condition\n\n\n");
      printf("  The Boundary Condition Number =");
      do{
        scanf("%d",&BCN);
       }while((BCN>3)||(BCN<0));
      if(!BCN){
              printf("The Translational and Rotational Stiffness:\n");
              printf("Beta=");
              scanf("%f",&beta);
              printf("Gama=");
              scanf("%f",&gama);
             }
    }
```

void appbdcon(lamda)

```
      float lamda;
    {
```

```
float A1 , A2 , A3 ;
if(!BCN){
          pointer1=&W1;
          A1=D1EI(0.0);
          D2W(0.0,pointer1);
          pointer1=&W2;
          A2=EI(0.0);
          D3W(0.0,pointer1);
          pointer1=&W3;
          D1W(0.0,pointer1);
          pointer1=&W4;
          W(0.0,pointer1);
          A3=FX(0.0);
          mat[0][0]=A1*W1.C0+A2*W2.C0-eta*eta*
                    A3*W3.C0+beta*W4.C0;
          mat[0][1]=A1*W1.C1+A2*W2.C1-eta*eta*
                    A3*W3.C1+beta*W4.C1;
          mat[0][2]=A1*W1.C2+A2*W2.C2-eta*eta*
                    A3*W3.C2+beta*W4.C2;
          mat[0][3]=A1*W1.C3+A2*W2.C3-eta*eta*
                    A3*W3.C3+beta*W4.C3;
          }

if(BCN<=1){
          pointer1=&W1;
          D2W(0.0,pointer1);
          A1=EI(0.0);
          pointer1=&W2;
          D1W(0.0,pointer1);
          mat[1][0]=A1*W1.C0-gama*W2.C0;
          mat[1][1]=A1*W1.C1-gama*W2.C1;
          mat[1][2]=A1*W1.C2-gama*W2.C2;
          mat[1][3]=A1*W1.C3-gama*W2.C3;
          }

          pointer1=&W1;
          A1=EI(1.0);
     if(A1==0.0)A1=1.0;
          D3W(1.0,pointer1);
          pointer1=&W2;
          A2=D1EI(1.0);
          D2W(1.0,pointer1);
          pointer1=&W3;
          D1W(1.0,pointer1);
          A3=FX(1.0);
          pointer1=&W4;
          W(1.0,pointer1);
          mat[2][0]=A1*W1.C0+A2*W2.C0-eta*eta*A3*W3.C0+
                    lamda*mew*W4.C0;
```

```
        mat[2][1]=A1*W1.C1+A2*W2.C1-eta*eta*A3*W3.C1+
                lamda*mew*W4.C1;
        mat[2][2]=A1*W1.C2+A2*W2.C2-eta*eta*A3*W3.C2+
                lamda*mew*W4.C2;
        mat[2][3]=A1*W1.C3+A2*W2.C3-eta*eta*A3*W3.C3+
                lamda*mew*W4.C3;

        pointer1=&W1;
        D2W(1.0,pointer1);
        A1=EI(1.0);
        if(A1==0.0)A1=1.0;
        mat[3][0]=A1*W1.C0;
        mat[3][1]=A1*W1.C1;
        mat[3][2]=A1*W1.C2;
        mat[3][3]=A1*W1.C3;

    }
```

float det()

```
    {
      int i , j , k , l=1 ;
      double s ;
      for(i=BCN;i<3;i++){
          if(!mat[i][i]){
                        j=i+1;
                        do{
                            if(mat[j][i]){
                                for(k=i;k<=3;k++){
                                    s=mat[i][k];
                                    mat[i][k]=mat[j][k];
                                    mat[j][k]=s;
                                           }
                                    l*=-1;
                                   }
                              else j+=1;
                          }while((j<=3)&&(!mat[i][i]));
                     }

          if(mat[i][i]){
                      for(j=i+1;j<=3;j++){
                          s=mat[j][i]/mat[i][i];
                          for(k=i;k<=3;k++)mat[j][k]-=s*mat[i][k];
                                        }
                      }
                    }
```

```
{
   float b , b1 , b2 ;
   float d , d1 , d2 ;

   b=RTINTE[n].x0;
   b2=RTINTE[n].x1;
   init();
   coef(b*b);
   appbdcon(b*b);
   d=det();
   init();
   coef(b2*b2);
   appbdcon(b2*b2);
   d2=det();
   do{
     b1=(b+b2)/2;
     init();
     coef(b1*b1);
     appbdcon(b1*b1);
     d1=det();
     if(d1*d2<0.0){
                    d=d1;
                    b=b1;
                   }
     if(d1*d<0.0){
                   d2=d1;
                   b2=b1;
                  }
      if(d1*d==0){
                   b2=b1;
                   b=b1;
                  }
           }while(fabs(b-b2)>EPS);
   w[n]=b;

}
```

void FUNC(n)

```
   int n ;
{
   int i , j , k ;
   float s,b[4];

                        s=w[n];
                        init();
                        coef(s*s);
                        appbdcon(s*s);
```

```
        s=1.0;
        for(i=BCN;i<=3;i++)s*=mat[i][i];
        return(l*s);
    }
void search()
    {
        int n=0;
        double s=0.000 , de1=0 , de2 ;
          init();
          coef(s*s);
          appbdcon(s*s);
          de1=det();
        if(de1==0.0){
                        RTINTE[n].x0=s;
                        RTINTE[n].x1=s;
                        n+=1;
                        s=0.5;
                        init();
                        coef(s*s);
                        appbdcon(s*s);
                        de1=det();
                   }
          s=1.0;
        do{
          init();
          coef(s*s);
          appbdcon(s*s);
          de2=det();
          if(de1*de2<0){
                        RTINTE[n].x0=s-1;
                        RTINTE[n].x1=s;
                        de1=de2;
                        n+=1;
                       }
          if(de1*de2==0.0){
                        RTINTE[n].x0=s-0.1;
                        RTINTE[n].x1=s+0.1;
                        de1=-de1;
                        n+=1;
                       }

          s+=1.0;
          }while(n<NEV);
    }


void root(n)
        int n ;
```

```
        for(i=0;i<BCN;i++)b[i]=0.0;
        b[BCN]=1.0;
        for(i=BCN+1;i<=3;i++)b[i]=-mat[i][BCN];
        for(i=BCN+1;i<3;i++){
                for(j=i+1;j<=3;j++){
                        s=mat[j][i]/mat[i][i];
                        for(k=i;k<=3;k++)mat[j][k]-=s*mat[i][k];
                        b[j]-=s*b[i];
                        }
                }
        b[3]=b[3]/mat[3][3];
        for(i=2;i>BCN;i--){
                s=0.0;
                for(j=3;j>i;j--)s+=b[j]*mat[i][j];
                b[i]=(b[i]-s)/mat[i][i];
                }
        s=0.0;
        for(i=0;i<=FORD;i++){
                F[i]=b[0]*c[i].C0+b[1]*c[i].C1+
                        b[2]*c[i].C2+b[3]*c[i].C3;
                }
        for(i=0;i<=FORD;i++)if(fabs(F[i])>fabs(s))s=F[i];
        for(i=0;i<=FORD;i++)F[i]/=s;
        for(i=0;i<=FORD;i++)K[n][i]=F[i];

    }
```

void disply()

```
    {
        int i,j,k,m,e;
        float a;
        a=eta*sin(BI*theta/180);
        printf(" Natural Frequenceis of a Rotating Beam With Non Uniform\n");
        printf("    Cross Section by The Series (Frobinius) Method\n\n\n");
        printf("______________________________________________________\n\n"
);
        printf("for:\n");
        printf("The Order of The W Function (FORD) = %d\n",FORD);
        printf("The Tolerance      (EPS)          = %f\n",EPS);
        printf("     Eta                          = %f\n",eta);
        printf("     Root Offset (Alpha)          = %f \n",alpha);
        printf("     Tip Mss (Mew)                = %f \n",mew);
        printf("     Setting Angle (Theta) in deg.= %f\n",theta);
        if(BCN==1)printf("     With Hinged Root Condition .\n");
        if(BCN==2)printf("     With Fixed Root Condition .\n");
        if(BCN==0){
```

```
            printf("    With Flexible Attachment with:\n");
            printf("         Beta = %f\n",beta);
            printf("         Gama = %f\n",gama);
            }
    printf("____________________________________________________________\n\n"
);
    printf("  The Coefficients of the g(x) and Flexural Rigidity f(x)\n");
    printf(" series with A(i) and B(i) are :-\n");
    printf("  i   A(i)     B(i) \n");
    printf("___________________________________________________________\n");
    for(i=0;i<=SORD;i++)printf("  %d   %e   %e\n",i,A[i],B[i]);
    printf("___________________________________________________________\n\n");
    m=NEV/5;
    if(NEV%5)m+=1;
    printf("Press Enter to Continue \n");
    getchar();
    getchar();
    for(i=1;i<=m;i++){
    if(i==m)e=NEV;
    else e=5*i;
    printf("\n\n\n   Eigen Values and Eigen Vectors\n");
    printf("              Table(%d)\n",i);
    printf("___________________________________________________________\n");
    printf("w    |");
    for(j=5*(i-1);j<=e-1;j++)printf("  w%d=     ",j+1);
    printf("\n__________________________________________________________\n");
    printf("   i |");
    for(j=5*(i-1);j<e;j++)printf("%8.5f  ",sqrt(w[j]*w[j]-a*a));
    printf("\n__________________________________________________________\n");
    for(k=0;k<=FORD;k++){
    printf(" %2d  |",k+1);
    for(j=5*(i-1);j<=e-1;j++)printf(" %9.8f ",K[j][k]);
    printf("\n");
    if(!((k+1)%10)){
            printf("Press Enter to Continue \n");
            getchar();
            }
                        }
    printf("____________________________________________________________\n")
;

                }
  }


void save()
 {
```

```
int i,j,k,m,e;
float a;
char Name[11];
FILE * fp;

printf("Enter The Output File Name (10 Chara.):\n");
getchar();
gets(Name);
if((fp=fopen(Name,"w"))==NULL){
puts("Cannot Open Output File\n");
exit(0);
                                }
a=eta*sin(BI*theta/180);
fprintf(fp," Natural Frequenceis of a Rotating Beam With Non Uniform\n");
fprintf(fp,"    Cross Section by the Series (Frobinius) Method\n\n\n");
fprintf(fp,"____________________________________________________________\n\n");
fprintf(fp,"for:\n");
fprintf(fp,"The Order of The W Function (FORD) = %d\n",FORD);
fprintf(fp,"The Tolerance     (EPS)        = %f\n",EPS);
fprintf(fp,"     Eta                       = %f\n",eta);
fprintf(fp,"     Root Offset (Alpha)       = %f \n",alpha);
fprintf(fp,"     Tip Mss (Mew)             = %f \n",mew);
fprintf(fp,"     Setting Angle (Theta) in deg.= %f\n",theta);
if(BCN==1)fprintf(fp,"     With Hinged Root Condition .\n");
if(BCN==2)fprintf(fp,"     With Fixed Root Condition .\n");
if(BCN==0){
fprintf(fp,"    With Flexible Attachment with:\n");
fprintf(fp,"          Beta = %f\n",beta);
fprintf(fp,"          Gama = %f\n",gama);
          }
fprintf(fp,"____________________________________________________________\n\n");
fprintf(fp,"  The Coefficients of the g(x) and Flexural Rigidity f(x)\n");
fprintf(fp," series with A(i) and B(i) are :-\n");
fprintf(fp,"  i   A(i)    B(i) \n");
fprintf(fp,"____________________________________________________________\n");
for(i=0;i<=SORD;i++)fprintf(fp,"  %d   %ef   %e\n",i,A[i],B[i]);
fprintf(fp,"____________________________________________________________\n\n");
m=NEV/5;
if(NEV%5)m+=1;
for(i=1;i<=m;i++){
if(i==m)e=NEV;
else e=5*i;
fprintf(fp,"\n\n\n   Eigen Values and Eigen Vectors\n");
fprintf(fp,"             Table(%d)\n",i);
fprintf(fp,"____________________________________________________________\n");
fprintf(fp,"   w |");
for(j=5*(i-1);j<=e-1;j++)fprintf(fp,"  w%d=     ",j+1);
fprintf(fp,"\n____________________________________________________________\n");
fprintf(fp,"   i |");
```

```
for(j=5*(i-1);j<=e-1;j++)fprintf(fp,"%8.5f ",sqrt(w[j]*w[j]-a*a));
fprintf(fp,"\n________________________________________________\n");
for(k=0;k<=FORD;k++){
fprintf(fp," %2d |",k+1);
for(j=5*(i-1);j<=e-1;j++)fprintf(fp," %9.8f ",K[j][k]);
fprintf(fp,"\n");
                    }
fprintf(fp,"________________________________________________\n");
                    }
 fclose(fp);
}
```

```
void Mdsh(){
        int i , j , NMS ;
        float dx , Fx , x , s ;
        float T[100][10] ;
        char Name[11];
        FILE * fp1;
        printf("Enter The Output File Name (10 Chara./.MSH/)\n");
        getchar();
        gets(Name);
        if((fp1=fopen(Name,"w"))==NULL){
        puts("Cannot Open Output File\n");
        exit(0);
                                        }
        printf("Enter The Number of Mode Shapes Required(NMS<=%d)=",NEV);
        scanf("%d",&NMS);
        printf("Enter the increment dx:");
        scanf("%f",&dx);
        x=0.0;
        while(x<=1.0){
        printf("%4.2f",x);
        fprintf(fp1,"%4.2f",x);
        for(j=0;j<NMS;j++){
                s=Fun(1.0,j);
                printf(" %6.4f",Fun(x,j)/s);
                fprintf(fp1," %6.4f",Fun(x,j)/s);
                    }
       printf("\n");
       fprintf(fp1,"\n");
       x+=dx;
           };
       fclose(fp1);

       }
```

```
void frme()
    {
      int i ;

      free(F);
      free(c);
      free(A);
      free(B);
      free(w);
      free(RTINTE);
      for(i=0;i<=NEV+1;i++)free(K[i]);
      free(K);
    }
```

4.3 Sapmle Run

Natural Frequenceis of a Rotating Beam With Non Uniform Cross Section by the Series (Frobinius) Method

---

for:
The Order of The W Function (FORD) = 35
The Tolerance (EPS) = 0.000001
Eta = 1.000000
Root Offset (Alpha) = 0.000000
Tip Mss (Mew) = 0.000000
Setting Angle (Theta) in deg .= 0.000000
With Fixed Root Condition

The Coefficients of the g(x) and Flexural Rigidity f(x) series with A(i) and B(i) are :-

i	A(i)	B(i)
0	1.000000e+00	1.000000e+00
1	-1.000000e+00	-1.000000e+00
2	5.000000e-01	5.000000e-01
3	-1.666670e-01	-1.666670e-01
4	4.166667e-02	4.166667e-02
5	-8.333340e-03	-8.333340e-03
6	1.388889e-03	1.388889e-03
7	-1.941270e-04	-1.941270e-04
8	2.481590e-05	2.481590e-05
9	-2.755730e-06	-2.755730e-06
10	2.755730e-07	2.755730e-07
11	-2.505211e-08	-2.505211e-08
12	2.087680e-09	2.087680e-09
13	-1.605900e-10	-1.605900e-10
14	1.147070e-11	1.147070e-11
15	-7.647200e-13	-7.647200e-13
16	4.779477e-14	4.779477e-14
17	-2.811457e-15	-2.811457e-15
18	1.561921e-16	1.561921e-16

Eigen Values and Eigen Vectors

Table(1)

w |	w1=	w2=	w3=	w4=
i |	4.86319	24.31400	63.96942	122.70290
1 |	0.00000000	0.00000000	0.00000000	0.00000000
2 |	0.00000000	0.00000000	0.00000000	0.00000000
3 |	1.00000000	0.63508517	0.08886536	0.00814221
4 |	-0.17107193	-0.74696618	-0.19307366	-0.02604150
5 |	-0.14684917	-0.41242221	-0.10198545	-0.01351997
6 |	-0.04804223	-0.13469259	-0.03330003	-0.00441412
7 |	0.04493086	1.00000000	1.00000000	0.33920902
8 |	0.00907656	-0.23473351	-0.65348095	-0.36966074

9	0.00023431	-0.21500210	-0.42443410	-0.21797168
10	-0.00002462	-0.06829555	-0.12988394	-0.06572239
11	0.00010703	0.10159819	0.78462088	1.00000000
12	0.00000583	0.00122238	-0.19351327	-0.51998758
13	-0.00000644	-0.01062677	-0.18180254	-0.36792928
14	-0.00000203	-0.00361891	-0.05646637	-0.11018321
15	-0.00000052	0.00188661	0.12475714	0.61007357
16	-0.00000014	0.00024600	-0.00747918	-0.15715352
17	-0.00000003	-0.00011311	-0.01813775	-0.14783074
18	-0.00000000	-0.00004328	-0.00592920	-0.04523546
19	0.00000000	0.00001011	0.00621827	0.11966205
20	0.00000000	0.00000219	0.00030814	-0.01282351
21	0.00000000	-0.00000040	-0.00061032	-0.02054312
22	0.00000000	-0.00000017	-0.00021608	-0.00653888
23	0.00000000	0.00000002	0.00012355	0.00961985
24	0.00000000	0.00000001	0.00001573	-0.00008160
25	0.00000000	-0.00000000	-0.00000857	-0.00122405
26	0.00000000	-0.00000000	-0.00000338	-0.00041154
27	0.00000000	0.00000000	0.00000114	0.00036974
28	-0.00000000	0.00000000	0.00000022	0.00002353
29	-0.00000000	-0.00000000	-0.00000006	-0.00003595
30	-0.00000000	-0.00000000	-0.00000003	-0.00001297
31	-0.00000000	0.00000000	0.00000001	0.00000756
32	-0.00000000	0.00000000	0.00000000	0.00000092
33	-0.00000000	-0.00000000	-0.00000000	-0.00000057
34	-0.00000000	-0.00000000	-0.00000000	-0.00000023
35	-0.00000000	-0.00000000	0.00000000	0.00000009
36	-0.00000000	-0.00000000	0.00000000	0.00000002

Appendix C

Numerical Results For The Parametric Study

The following tables show the results which was found from the parameteric study for rotating, exponintially varying cross-section beam considered in chapter five.

The effect of the non-dimentional rotational speed on the natural frequencies for fixed root beam with $\mu=0$, $\theta=0$, $\alpha=0$ using the series method with thirty five approximating function order and toleranc of 1.0×10^{-6}

η	w1=	w2=	w3=	w4=
1	4.86319	24.31400	63.96942	122.70290
2	5.22819	24.64763	64.30175	123.06821
3	5.78262	25.19411	64.83343	123.75780
4	6.47453	25.94003	65.58059	124.67697
5	7.26109	26.86901	66.52200	125.55859
6	8.11150	27.96246	67.65405	127.12704
7	9.00488	29.20190	68.97571	128.48633
8	9.92737	30.56903	70.45703	130.12720
9	10.86993	32.04688	72.09467	132.03723
10	11.82660	33.62134	73.87885	133.60840

Table (C.1)

The effect of the non-dimentional rotational speed on the natural frequencies for hinged root beam with $\mu=0$, $\theta=0$, $\alpha=0$ using the series method with thirty five approximating function order and toleranc of 1.0×10^{-6}

η	w1=	w2=	w3=	w4=
1	1.00000	16.66604	51.23320	105.49380
2	2.00000	17.12127	51.62280	105.87267
3	3.00000	17.85397	52.26455	106.51424
4	4.00000	18.83135	53.15161	107.39137
5	5.00000	20.01704	54.26929	108.46776
6	6.00000	21.37600	55.60448	109.77684
7	7.00000	22.87671	57.13712	111.30105
8	8.00000	24.49276	58.85590	113.07097
9	9.00000	26.20213	60.74318	114.97165
10	10.00000	27.98819	62.77653	117.12868

Table (C.2)

The effect of the the tip mass ratio on the natural frequencies for fixed root beam with $\eta=1$, $\theta=0$, $\alpha=0$ using The series method with thirty five approximating function order and toleranc of 1.0×10^{-6}

μ	w1=	w2=	w3=	w4=
0.00	4.86319	24.31400	63.96942	122.70290
0.05	4.05650	20.82025	56.46935	111.23795
0.10	3.56526	19.51972	54.51868	108.93931
0.15	3.23023	18.86135	53.65989	108.01453
0.20	2.98465	18.47255	53.18945	107.53349
0.25	2.79553	18.22151	52.89948	107.25000
0.30	2.64463	18.05018	52.70873	107.06297
0.35	2.52093	17.92905	52.57961	106.88973
0.40	2.41737	17.84150	52.48836	106.83817
0.45	2.32919	17.77763	52.42382	106.76551
0.50	2.25305	17.73090	52.37988	106.71313

Table(C.3)

The effect of the the tip mass ratio on the natural frequencies for Hinged root beam with $\eta=1$, $\theta=0$, $\alpha=0$ using The series method with thirty five approximating function order and toleranc of 1.0×10^{-6}

μ	w1=	w2=	w3=	w4=
0.00 i	1.00000	16.66604	51.23320	105.4938
0.05 i	1.00000	14.05514	44.81767	94.81250
0.10 i	1.00000	12.94313	42.99358	92.54008
0.15 i	1.00000	12.34050	42.16833	91.60959
0.20 i	1.00000	11.97025	41.71077	91.10153
0.25 i	1.00000	11.72541	41.42864	90.81349
0.30 i	1.00000	11.55594	41.24310	90.62532
0.35 i	1.00000	11.43530	41.11711	90.49621
0.40 i	1.00000	11.34810	41.02959	90.40783
0.45 i	1.00000	11.28478	40.96861	90.34026
0.50 i	1.00000	11.23911	40.92707	90.30859

Table (C.4)

The effect of the the root offset ratio on the natural frequencies for fixed root beam with $\eta=1$, $\mu=0$, $\theta=0$ using The series method with thirty five approximating function order and toleranc of 1.0×10^{-6}

α	w1=	w2=	w3=	w4=
0.0	4.86319	24.31400	63.96942	122.70290
0.5	4.94839	24.39092	64.04899	122.80781
1.0	5.03211	24.46771	64.12950	123.00075
1.5	5.11446	24.54414	64.20647	123.00393
2.0	5.19546	24.62027	64.28173	123.01470
2.5	5.27520	24.69616	64.36426	123.13117
3.0	5.35374	24.77199	64.43353	123.21386
3.5	5.43111	24.84750	64.51919	123.30788
4.0	5.50739	24.92272	64.58971	123.34371
4.5	5.58259	24.99767	64.66796	123.54337
5.0	5.65678	25.07226	64.75086	123.65626

Table (C.5)

The effect of the the root offset ratio on the natural frequencies for hinged root beam with $\eta=1$, $\mu=0$, $\theta=0$ using The series method with thirty five approximating function order and toleranc of 1.0×10^{-6}

α	w1=	w2=	w3=	w4=
0.0	1.00000	16.66604	51.23320	105.49380
0.5	1.34985	16.78226	51.33462	105.56249
1.0	1.62576	16.89771	51.43250	105.68066
1.5	1.86094	17.01227	51.53344	105.77245
2.0	2.06931	17.12605	51.63437	105.86718
2.5	2.25832	17.23901	51.73369	105.99312
3.0	2.43250	17.35120	51.83247	106.08898
3.5	2.59484	17.46258	51.93160	106.18793
4.0	2.74744	17.57325	52.02969	106.28178
4.5	2.89185	17.68323	52.12889	106.35923
5.0	3.02925	17.79241	52.22868	106.48089

Table(C.6)

The effect of the the setting angle on the natural frequencies for fixed root beam with $\eta=1$, $\mu=0$, $\alpha=0$ using The series method with thirty five approximating function order and toleranc of 1.0×10^{-6}

θ	w1=	w2=	w3=	w4=
.00	4.86319	24.31400	63.96942	122.70290
.10	4.86009	24.31338	63.96919	122.70278
.20	4.85115	24.31159	63.96851	122.70243
.30	4.83742	24.30885	63.96747	122.70189
.40	4.82052	24.30550	63.96619	122.70122
.50	4.80248	24.30192	63.96483	122.70051
.60	4.78546	24.29857	63.96356	122.69985
.70	4.77154	24.29583	63.96252	122.69931
.80	4.76243	24.29404	63.96184	122.69895
.90	4.75927	24.29342	63.96160	122.69883

Table (C.7)

The effect of the the setting angle on the natural frequencies for hinged root beam with $\eta=1$, $\mu=0$, $\alpha=0$ using The series method with thirty five approximating function order and toleranc of 1.0×10^{-6}

θ	w1=	w2=	w3=	w4=
00.0	1.00000	16.66604	51.23320	105.49380
10.0	0.98481	16.66513	51.23291	105.49366
20.0	0.93969	16.66253	51.23206	105.49325
30.0	0.86603	16.65854	51.23076	105.49262
40.0	0.76604	16.65364	51.22917	105.49185
50.0	0.64279	16.64842	51.22747	105.49102
60.0	0.50000	16.64352	51.22588	105.49025
70.0	0.34202	16.63953	51.22458	105.48962
80.0	0.07365	16.63692	51.22373	105.48921
90.0	0.00000	16.63601	51.22344	105.48907

Table (C.8)

The effect of the the setting angle on the natural frequencies for fixed root beam with $\eta=5$, $\mu=0$, $\alpha=0$ using The series method with thirty five approximating function order and toleranc of 1.0×10^{-6}

θ	w1=	w2=	w3=	w4=
00.0	7.26109	26.86901	66.52200	125.55859
10.0	7.20900	26.85498	66.51634	125.55558
20.0	7.05684	26.81454	66.50002	125.54694
30.0	6.81715	26.75245	66.47501	125.53369
40.0	6.51107	26.67610	66.44432	125.51745
50.0	6.16870	26.59461	66.41164	125.50015
60.0	5.82868	26.51780	66.38092	125.48390
70.0	5.53606	26.45502	66.35587	125.47065
80.0	5.33641	26.41397	66.33951	125.46200
90.0	5.26531	26.39969	66.33383	125.45899

Table (C.9)

The effect of the translation stiffness ratio on the natural frequencies for exponintially varying cross-section properties with η=1, μ=0, θ=0, α=0 $\gamma=1.0\times10^{11}$. using The series method with thirt; five approximating function order and toleranc of 1.0×10^{-6}

β	w1=	w2=	w3=	w4=
0	0.00000	6.86351	31.55711	75.98086
1	1.23594	6.97591	31.58875	76.01797
2	1.71687	7.09298	31.62024	76.02498
3	2.06476	7.21430	31.65163	76.02296
4	2.34058	7.33939	31.68341	76.03093
5	2.56861	7.46771	31.71534	76.03598
6	2.76173	7.59876	31.74701	76.06252
7	2.92786	7.73201	31.77917	76.06152
8	3.07238	7.86696	31.81155	76.07810
9	3.19920	8.00313	31.84364	76.07812
10	3.31124	8.14009	31.87611	76.11095
20	3.96510	9.48905	32.20773	76.24521
30	4.24555	10.71040	32.55139	76.37498
40	4.39597	11.78328	32.90591	76.51656
50	4.48853	12.72597	33.27057	76.65049
60	4.55089	13.55917	33.64447	76.79345
70	4.59563	14.30020	34.02671	76.93446
80	4.62923	14.96282	34.41551	77.07665
90	4.65540	15.55817	34.81066	77.22208
100	4.67632	16.09514	35.20994	77.36697
200	4.77023	19.43782	39.24166	78.90853
300	4.80136	20.96759	42.90356	80.60513
400	4.81687	21.79881	45.97998	82.38964
500	4.82616	22.30947	48.49693	84.25974
600	4.83235	22.65167	50.54307	86.17189
700	4.83677	22.89576	52.20552	88.05875
800	4.84008	23.07824	53.56472	89.93291
900	4.84265	23.21954	54.68227	91.71872
1000	4.84471	23.33198	55.61121	93.45264
2000	4.85396	23.83128	59.97241	105.75706
3000	4.85704	23.99413	61.38915	111.65784
4000	4.85857	24.07499	62.07555	114.72797
5000	4.85950	24.12315	62.47510	116.56549
6000	4.86012	24.15523	62.73801	117.63854

7000	4.86056	24.17809	62.91821	118.51800
8000	4.86089	24.19518	63.05273	119.15726
9000	4.86114	24.20852	63.15794	119.49924
10000	4.86134	24.21908	63.24120	119.74657
20000	4.86227	24.26662	63.61382	121.31543
30000	4.86258	24.28247	63.73261	121.93752
40000	4.86273	24.29037	63.79103	122.07337
50000	4.86282	24.29510	63.82812	122.20996
60000	4.86288	24.29829	63.85320	122.30470
70000	4.86293	24.30057	63.86638	122.38024
80000	4.86296	24.30209	63.88156	122.38914
90000	4.86298	24.30346	63.88854	122.40415
100000	4.86301	24.30453	63.89868	122.42962
1e6	4.86317	24.31297	63.96093	122.70215
1e7	4.86319	24.31387	63.96942	122.70215
1e8	4.86319	24.31400	63.96942	122.70290
1e9	4.86319	24.31400	63.96942	122.70290
Infin	4.86319	24.31400	63.96942	122.70290

Table(C.10)

The effect of the rotational stiffness ratio on the natural frequencies for exponintially varying cross-section properties $\eta=1$, $\mu=0$, $\theta=0$, $\alpha=0$ $\beta=1.0\times10^{11}$. using The series method with thirty five approximating function order and toleranc of 1.0×10^{-6}

γ	w1=	w2=	w3=	w4=
0	1.00000	16.66604	51.23320	105.49380
1	2.43392	17.69214	52.28601	106.56445
2	3.02083	18.48701	53.19470	107.46485
3	3.37076	19.11762	53.98199	108.37500
4	3.60673	19.62854	54.67117	109.12500
5	3.77748	20.05017	55.27666	109.71868
6	3.90705	20.40356	55.81129	110.37499
7	4.00883	20.70367	56.28711	110.95309
8	4.09096	20.96174	56.71223	111.51621
9	4.15863	21.18584	57.09554	112.03422
10	4.21538	21.38213	57.44090	112.47796
20	4.50420	22.51637	59.64434	115.53806
30	4.61486	23.01940	60.74572	117.13378

40	4.67337	23.30279	61.40528	118.28515
50	4.70956	23.48455	61.83817	118.95867
60	4.73417	23.61089	62.14756	119.48673
70	4.75198	23.70388	62.37967	119.89172
80	4.76546	23.77534	62.55745	120.25598
90	4.77604	23.83154	62.70412	120.44537
100	4.78455	23.87722	62.82129	120.65896
200	4.82339	24.08971	63.37415	121.64095
300	4.83656	24.16309	63.56694	122.07326
400	4.84317	24.20031	63.66203	122.20945
500	4.84716	24.22287	63.72483	122.30469
600	4.84982	24.23795	63.76968	122.37695
700	4.85172	24.24879	63.79218	122.38035
800	4.85315	24.25688	63.81457	122.42957
900	4.85426	24.26315	63.83313	122.42962
1E3	4.85516	24.26819	63.84718	122.43739
1E4	4.86239	24.30946	63.95403	122.70094
1E5	4.86311	24.31349	63.96933	122.70936
1E6	4.86318	24.31400	63.96942	122.70290
1E7	4.86319	24.31400	63.96942	122.70290
1E8	4.86319	24.31400	63.96942	122.70290
1E9	4.86319	24.31400	63.96942	122.70290
1E10	4.86319	24.31400	63.96942	122.70290
1E11	4.86319	24.31400	63.96942	122.70290
INFI	4.86319	24.31400	63.96942	122.70290

Table (C.11)

ملخص باللغة العربية

الإهتزاز في العوارض الدوارة غير منتظمة المقطع العرضي

عمر عبد الكريم الشبلي

إشراف

الدكتور مازن القيسي

في هذا البحث تمّت دراسةُ الاهتزازِ الحرّ في العوارضِ الـدوّارةِ ذاتِ المقطعِ العرضـي غيرِ الثابتِ باستخدامِ طريقةِ المتسلسلاتِ الهندسيةِ. و قد أُستُخدِمَ نموذجُ أيلر البسيطِ لتمثيـلِ الحالةِ العامةِ لعارضةٍ دوّارةٍ ذاتِ مقطعٍ عرضي غيرِ ثـابتٍ، و ذاتِ أزاحـةٍ جذريـةٍ عـن محورِ الدورانِ مثبتةٍ إلى الجذرِ تثبيتاً مرناً و تحملُ كتلةً طرفيةً. و لاشتقاقِ معادلةِ الحركةِ و الشروطِ الحديةِ أُستُخدِمَ مبدأُ هاميلتون. ثم كُتِبت معادلةُ الحركةِ و الشروطِ الحديةِ بشكلٍ لا بعديٍ.

لقـد حُلَّـت معادلـةُ الحركـةِ الـتي تحكـمُ هـذا النظـامَ باسـتخدامِ طريقـةِ المتسلسـلاتِ الهندسيةِ و طريقةِ العنصرِ المحـدودِ. وقُورِنـت النتـائجُ الـتي تمّ الحُصـولُ عليهـا ببعـضِ النتـائجِ المنشورةِ في أبحاثٍ سابقةٍ. ثم استخدمت طريقةُ العنصرِ المحدودِ للتـأكدِ مـن صحـةِ النتـائجِ التي تمّ الحُصُولُ عليها بإستخدامِ طريقةِ المتسلسلاتِ الهندسيةِ للحالاتِ التي لم يتوفر لها نتـائجٌ منشورةٌ. ثم أُجريت دراسةٌ نظريةٌ للعواملِ التي تؤثـرُ علـى حـلِ هـذهِ المسـألةِ لعارضـةٍ دوّارةٍ ذاتِ مقطعٍ عرضيٍّ تتغيرُ خواصُّهُ أُسّياً. وقد دُرِسَ تأثيرُ كـلٍ مـن سـرعةِ الـدورانِ، و الكتلـةِ الطرفيةِ، و نصفِ قطرِ الإزاحةِ الجذريةِ، و زاويةِ التثبيـتِ بالإضافـةِ إلى تأثـيرِ مرونـةِ التثبيـتِ الانتقاليةِ و ألد ورانيةِ عندَ الجذرِ.

من هذه الدراسة وجِدَ أنَّ زيادةَ كلٍ من سرعةِ الدوران، نصفِ قطرِ الإزاحةِ الجذريةِ، و زاويةِ التثبيتِ، و مرونةِ التثبيتِ الانتقاليةِ و ألد ورانيةِ عندَ الجذرِ تؤدّي إلى زيادةِ قيمة التردُّداتِ الطبيعيةِ للعارضةِ، كذلك فانَّ زيادةَ كلٍ من زاويةِ التثبيتِ، و الكتلةِ الطرفيةِ يؤدّي إلى نقصانِ قيمةِ التردُّداتِ الطبيعيةِ. كما وجِدَ أنَّ طريقةَ العنصرِ المحدودِ تُعطي نتائجَ تتفقُ إلى حدٍّ كبيرٍ مع النتائجِ التي تعطيها طريقةُ المتسلسلاتِ الهندسيةِ. من كل هذا وجدَ أنَّ طريقةَ المتسلسلاتِ الهندسيةِ هي طريقةٌ فعّالةٌ إذا ما تمَّ برمجتُها.